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Introduction to Dubois–Violette's Noncommutative Differential Geometry

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We present a detailed review of the Dubois-Violette approach to noncommutative differential calculus. The noncommutative differential geometry of matrix algebras and the noncommutative Poisson structures are treated in some detail. We also present the analog of Maxwell's theory and new models of Yang–Mills–Higgs theories that can be constructed in this framework. In particular, some simple models are compared with the standard model. Finally, we discuss some perspectives and open questions.

1. INTRODUCTION AND PRELIMINARIES

Let M be a smooth manifold, $A_0 = C^\infty(M)$ the algebra of smooth complex functions on M, $C^0(M)$ the algebra of continuous functions on M, E a smooth complex vector bundle of finite rank over M, $\Gamma(E)$ the space of smooth sections of E, $V(M)$ the Lie algebra of complex vector fields over M, $\mathrm{Der}(A_0)$ the Lie algebra of derivations of A_0, $C_0^* = C(\mathrm{Der}(A_0); A_0)$ the Chevalley complex of cochains of $\mathrm{Der}(A_0)$ with values in A_0, and $\Omega(M)$ the graded differential algebra of differential forms on M.

In *commutative differential geometry*, if we consider the algebra A_0 as an abstract commutative associative $C^\star$-algebra, one may really study the manifold M using A_0. Effectively, everything concerning the geometry and topology of a complex smooth differentiable manifold M can be found out via investigation of all C-valued smooth functions over M. The choice of smooth functions here is not innocent since it avoids losing information about the differentiable structure of M, which is the case when we choose the

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algebra $C^0(M)$. In order to be sufficiently representative, the set $A_0 = C^\infty(M)$ has to possess some special properties, for instance, if we want to describe a Haussdorffian space, the following property must be satisfied:

$$\text{If } x_1 \neq x_2 \in M, \text{ then there exists at least one function } f \in A_0 / f(x_1) \neq f(x_2).$$

This guarantees the separability of the points in M. The set A_0 naturally forms an algebra: Any two functions $f_1, f_2 \in A_0$ can be multiplied by multiplying their values at the same point $x \in M$:

$$(f_1 . f_2)(x) = f_1(x) f_2(x)$$

The same holds for the multiplication by a scalar and for addition. The maximal ideals of this algebra can be put into one-to-one correspondence with the points of the manifold M. Let

$$I_x = \{f \in A_0 / f(x) = 0, x \in M\} \subset A_0$$

Then, for any $g \in A_0$, $g.f \in I_x$ if $f \in I_x$. This means that I_x is an *ideal* of A_0. It is easy to see that I_x is also a *maximal ideal* of A_0 since if we add any other element of A_0 to it, it ceases to be an ideal. Therefore, any point in M determines a maximal ideal in A_0. The inverse is also true (Kobayashi and Nomizu, 1963).

Moreover, A_0 possesses many derivations (i.e., the complex vector fields over M), contrary to $C^0(M)$. Indeed, if χ is a vector field over M, then

$$\chi: \quad A_0 \to A_0$$

$$\chi(k_1 f_1 + k_2 f_2) = k_1 \chi(f_1) + k_2 \chi(f_2) \qquad \text{(linearity)}$$

$$\chi(f . g) = \chi(f) . g + f . \chi(g) \qquad \text{(Leibnitz rule)}$$

It is also interesting to note that $\text{Der}(A_0)$ is a left A_0-module, i.e.,

$$\text{If } \chi \in \text{Der}(A_0), \text{ then } f\chi \in \text{Der}(A_0), \ \forall f \in A_0$$

In this context, $\Gamma(E)$ is a finite projective module over A_0 and the correspondence $E \to \Gamma(E)$ is an equivalence between the category of smooth complex vector bundles on M and the category of finite projective A_0-modules. Similarly, the notion of Hermitian vector bundle generalizes into a notion of Hermitian module for a $C^\star$-algebra (Connes, 1980). Moveover, the Lie algebra $V(M)$ coincides with the Lie algebra $\text{Der}(A_0)$.

Therefore, we also know by definition that the Lie algebra $\text{Der}(A_0)$ acts by derivations on A_0 and consequently that the space of all antisymmetric multilinear mappings from $\text{Der}(A_0)$ into A_0 (i.e., the Chevalley complex C_0^*) is a graded differential algebra. Then, we observe that the graded differential algebra $\Omega(M)$ of differential forms on M is just the smallest graded differential subalgebra $\Omega_{\text{Der}}(A_0)$ of C_0^* which contains $A_0 \equiv C^0(\text{Der}(A_0); A_0)$.

Finally, $\Omega_{\mathrm{Der}}(A_0)$ will serve to define a notion of connection on finite projective A_0-modules which corresponds to the usual notion of connection on vector bundles. Table I summarizes this information, which permits to define a *commutative differential calculus*.

The fact that the graded differential algebra $\Omega(M)$ of differential forms on M represents the smallest graded differential subalgebra $\Omega_{\mathrm{Der}}(A_0)$ of the complex $C_0^* = C(\mathrm{Der}(A_0); A_0)$ which contains A_0 led Dubois-Violette (1988, 1989) to propose a *noncommutative* generalization of the usual differential calculus presented above.

In *noncommutative differential geometry*, the role of the $C^\star$-algebra A_0 is played by an abstract *associative (not necessarily commutative)* $C^\star$-algebra A, as analog of functions on *noncommutative* spaces (Connes, 1986).

From this point of view, which originates from quantum mechanics (Heisenberg, 1925; Born and Jordan, 1925; Born *et al.*, 1926), where the Hamiltonian vector fields are replaced by derivations [i.e., the so-called *quantum differentiations* (Dirac, 1926)], it is natural to consider that the *noncommutative* generalization of the notion of vector field is that of derivation and that the analog of the differentiable structure is encoded in the Lie algebra of derivations. Moreover, it was realized very earlier that quantum mechanics should be described in the context of some noncommutative Poisson structure (Dirac, 1926).

In order to define connections on A-modules which generalize the notion of connections on vector bundles, and consequently to define a *noncommutative* differential calculus, we need to define a *noncommutative* generalization of the graded differential algebra of differential forms.

Following Connes' procedure, modules of sections of vector bundles generalize to finite projective A-modules. But, we remark that, contrary to the Lie algebra of vector fields on M, which is an A_0-module, the Lie algebra

Table I. Correspondence Table

Usual differential geometry	$C^\star$-algebraic geometry
Smooth differentiable manifold: M	Commutative associative $C^\star$-algebra: $A_0 = C^\infty(M)$
Lie algebra of complex vector fields over M: $V(M)$	Lie algebra of derivations of A_0: $\mathrm{Der}(A_0)$
Finite-rank smooth complex vector bundle over M: E	Finite projective A_0-module: $\Gamma(E)$
Graded differential algebra of differential forms on M: $\Omega(M)$	Smallest graded differential sub-algebra of the complex $C_0^* = C(\mathrm{Der}(A_0); A_0)$ containing A_0: $\Omega_{\mathrm{Der}}(A_0)$
Connection on E: ∇: $\chi \to \nabla_\chi$	Connection on $\Gamma(E)$: ∇: $\Gamma(E) \to \Gamma(E) \otimes \Omega^1_{\mathrm{Der}}(A_0)$

Der(A) of derivations of A is not an A-module in the *noncommutative* case. This will very possibly justify the existence of several *noncommutative* generalization procedures in the literature (Dubois-Violette, 1988, 1989; Connes, 1986; Karoubi, 1983).

This work is organized as follows. In Section 2, we present a detailed and self-consistent review of Dubois-Violette's approach to *noncommutative differential calculus*. In Section 3, we treat the simple case of the algebra $\mathbf{M}_n(C)$ of complex $n \times n$ matrices ($n \geq 2$), where we develop various concepts of differential geometry on $\mathbf{M}_n(C)$ using $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ as algebra of differential forms. Hence, we introduce the notions of volume element, of integration of differential forms, and of *closed graded trace*, define a canonical invariant Riemannian structure [the analogs of a metric for $\mathbf{M}_n(C)$ and the corresponding scalar product on $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$], and describe the corresponding Hodge theory on $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$. In particular, the algebra $\mathbf{M}_2(C)$ of 2×2 matrices is described in some detail. We also investigate the noncommutative differential geometry of the algebra $A = C^\infty(M) \otimes \mathbf{M}_n(C)$. In Section 4, we present Dubois-Violette's noncommutative generalization of the Poisson structures. In Section 5, we study the symplectic geometry of the algebra $\mathbf{M}_n(C)$ and show that there is a canonical invariant symplectic form $\omega \in \Omega^2_{\mathrm{Der}}(\mathbf{M}_n(C))$ for which the corresponding *generalized* Poisson bracket $\{\,,\}$ is given by

$$\{E, F\} = \frac{i}{\hbar}[E, F]$$

$\forall E, F \in \mathbf{M}_n(C)$. This shows clearly that quantum mechanics is a noncommutative symplectic geometry. We also study the cases of the Heisenberg algebra $A_\hbar$ and its matrix version. In Section 6, we complete the theoretical Dubois-Violette approach by studying the notions of gauge group, connections, and their associated curvatures on Hermitian A-modules. This leads us to discuss, in Section 7, the new models of gauge theory proposed by Dubois-Violette *et al.* in the context of their approach. Section 8 is devoted to some conclusions concerning the general formulation of this approach and its applications, and discussions of some problems and open questions.

An appendix deals with some technical points raised in the discussion of Section 5.2 on symplectic structure. Finally, we give a more or less complete list of references on the subject.

2. DUBOIS-VIOLETTE'S APPROACH

Let A be an associative (not necessarily commutative) $C^\star$-algebra with unit $\mathbb{1}$ and a pointwise product "$\cdot$" and let Der(A) be the Lie algebra of all derivations of A in itself:

$$\mathrm{Der}(A) = \{\chi \in \mathrm{End}(A) \,/\, \chi(a \cdot b) = \chi(a) \cdot b + a \cdot \chi(b), \forall a, b \in A\} \quad (1)$$

where the Lie bracket is the commutator in End(A). If M is an A-module, we add the rule

$$1 \cdot x = x \qquad \text{for} \quad x \in M \tag{2}$$

to the definition axioms of (left-) modules.

2.1. The Graded Algebra $\tau(A)$

Let $\tau^n(A)$ denote the space $A^{\otimes^{n+1}} = A \otimes A \otimes \cdots \otimes A$ ($n + 1$ factors), $n \in \mathbf{N}$.

These spaces are canonically A-bimodules and

$$\tau^n(A) \otimes_A \tau^m(A) = \tau^{n+m}(A)$$

It follows that

$$\tau(A) = \bigoplus_{n \in \mathbf{N}} \tau^n(A) \tag{3}$$

with $\tau^0(A) \equiv A$, is a graded algebra. The associated product μ

$$\mu\colon \quad \tau^n(A) \times \tau^m(A) \to \tau^{n+m}(A)$$

is defined by

$$\begin{aligned} \mu[(a_0 \otimes \cdots \otimes a_n), (b_0 \otimes \cdots \otimes b_m)] &= (a_0 \otimes \cdots \otimes a_n) \cdot (b_0 \otimes \cdots \otimes b_m) \\ &= a_0 \otimes \cdots \otimes a_{n-1} \otimes \\ &\quad \times (a_n \cdot b_0) \otimes b_1 \otimes \cdots \otimes b_m \end{aligned} \tag{4}$$

where a_i, $b_j \in A$.

One defines a linear mapping

$$d\colon \quad \tau^n(A) \to \tau^{n+1}(A)$$

by

$$\begin{aligned} d(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{p=0}^{n+1} (-1)^p a_0 \otimes \cdots \otimes a_{p-1} \otimes 1 \otimes a_p \otimes \\ \times \cdots \otimes a_n \end{aligned} \tag{5}$$

d is a nilpotent antiderivation of degree one. In addition to this differential, one defines boundary mappings $\tilde{b}_k$, $\tilde{b}$, c, and b: $\tau^n(A) \to \tau^{n-1}(A)$ with $n \geq 1$ and $0 \leq k \leq n - 1$, defined by

$$\begin{aligned} \tilde{b}_k(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_{k-1} \otimes (a_k \cdot a_{k+1}) \otimes a_{k+2} \otimes \\ \times \cdots \otimes a_n \end{aligned} \tag{6}$$

$$\tilde{b} = \sum_{k=0}^{n-1} (-1)^k \tilde{b}_k \tag{7}$$

$$c(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n (a_n \cdot a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1} \tag{8}$$

$$b = \tilde{b} + c \tag{9}$$

respectively. $\tilde{b}$ is a nilpotent antiderivation of degree -1 and one has

$$\tilde{b} \circ c + c \circ \tilde{b} = 0 \tag{10a}$$

$$b^2 = c^2 = 0 \tag{10b}$$

One may also define a derivation of degree zero. One has

$$d \circ \tilde{b} + \tilde{b} \circ d = 0 \tag{10c}$$

$$d \circ b + b \circ d = d \circ c + c \circ d$$

such that

$$\begin{aligned}(d \circ b + b \circ d)(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_n \\ &\quad - (-1)^{n-1} \mu[da_n, a_0 \otimes \cdots \otimes a_{n-1}] \\ &= a_0 \otimes \cdots \otimes a_n \\ &\quad - (-1)^{n-1} da_n \cdot (a_0 \otimes \cdots \otimes a_{n-1}) \end{aligned} \tag{10d}$$

where μ is defined by equation (4) and da_n is given by [see equation (5)]

$$da_n = \mathbb{1} \otimes a_n - a_n \otimes \mathbb{1}$$

Finally, one has the following result.

Proposition 1. The cohomology $H(\tau(A))$ of $\tau(A)$ is trivial, i.e., $H^0(\tau(A)) = C$ and $H^n(\tau(A)) = 0$ for $n \geq 1$.

Proof. Let

$$i: \quad C \to A$$

be a mapping defined by

$$i(k) = k\mathbb{1} \tag{11}$$

for all $k \in C$. One has

$$d \circ i = 0 \tag{12}$$

so one has a complex

$$0 \to C \xrightarrow{i} \tau^0(A) \xrightarrow{d} \tau^1(A) \xrightarrow{d} \cdots \xrightarrow{d} \tau^n(A) \xrightarrow{d} \tau^{n+1}(A) \xrightarrow{d} \cdots \tag{13}$$

with a differential d. Let $\omega \in A^\star$ be a linear form on A satisfying

$$\omega(\mathbb{1}) = 1 \tag{14}$$

To prove Proposition 1, one must define a contracting homotopy k_ω for the above complex. To reach this goal, one defines k_ω by

$$k_\omega(C) = 0 \tag{15a}$$

$$k_\omega(a_0 \otimes \cdots \otimes a_n) = \omega(a_0)a_1 \otimes a_2 \otimes \cdots \otimes a_n \qquad \text{for} \quad n \geq 0 \tag{15b}$$

Then, k_ω is a contracting homotopy for the above complex since one has

$$d \circ k_\omega + k_\omega \circ d = \mathrm{Id}_{T^n(A)} \quad \blacksquare \tag{16}$$

2.2. The Graded Differential Algebra $\Omega(A)$

We will consider the definition of $\Omega(A)$ as that given in Karoubi (1982, 1983). Let

$$m: \quad A \otimes A \to A$$

be the product on A such that $\forall a, b \in A$

$$m(a \otimes b) = a \cdot b \tag{17}$$

and $\otimes$ is the topological tensor product. One defines an A-bimodule $\Omega^1(A)$ [$\Omega^1(A)$ is a subbimodule of $A \otimes A$] by

$$\Omega^1(A) = \mathrm{Ker}(m) \tag{18}$$

and a derivation d of A with values in $\Omega^1(A)$ by

$$d: \quad A \to \Omega^1(A)$$

such that

$$da = \mathbb{1} \otimes a - a \otimes \mathbb{1} \tag{19}$$

Recall that a derivation of A in a A-bimodule M is a linear mapping

$$\delta: \quad A \to M$$

satisfying

$$\delta(a \cdot b) = \delta(a) \cdot b + a \cdot \delta(b) \tag{20}$$

$\forall a, b \in A$. Then, $(\Omega^1(A),d)$ is characterized by the following universal property (Cartan and Eilenberg, 1973; Bourbaki, 1970).

Property 1. For each derivation δ

$$\delta: \quad A \to M$$

of A with values in an A-bimodule M, there exists a unique homomorphism (up to an isomorphism) of A-bimodules:

$$i_\delta: \quad \Omega^1(A) \to M$$

such that

$$\delta = i_\delta \circ d$$

This property comes from the fact that $\Omega^1(A)$ is generated as left (or right) A-module by dA.

Let us take

$$\Omega^0(A) = A$$

and

$$\Omega^n(A) = \underbrace{\Omega^1(A) \otimes_A \cdots \otimes_A \Omega^1(A)}_{n \text{ factors } (n \geq 1)} \tag{21}$$

Then, a graded differential algebra naturally results:

$$\Omega(A) = \bigoplus_{n \in \mathbf{N}} \Omega^n(A) \tag{22}$$

Furthermore, we have the following result.

Proposition 2. The derivation

$$d: \quad A = \Omega^0(A) \to \Omega^1(A)$$

extends uniquely in a differential of $\Omega(A)$, i.e., in a nilpotent antiderivation of degree one of $\Omega(A)$ also denoted by d.

$\Omega^n(A)$ is a submodule of $\tau^n(A)$ and one easily verifies that $(\Omega(A), d)$ is a graded differential subalgebra of $(\tau(A), d)$ defined in Section 2.1.

It follows from Property 1 that $\Omega(A)$ is characterized by the following (Connes, 1986; Karoubi, 1982):

Property 2. Any homomorphism

$$\phi: \quad A \to \tilde{\Omega}^0$$

of unital algebras, where $(\tilde{\Omega} = \oplus_{n \in \mathbf{N}} \tilde{\Omega}^n, d')$ is a graded differential algebra, lifts uniquely as a homomorphism of graded differential algebras:

$$\Phi: \quad (\Omega(A), d) \to (\tilde{\Omega}, d')$$

It is evident from Section 2.1 that

$$\Omega^n(A) = \bigcap_{k=0}^{n-1} \mathrm{Ker}(\tilde{b}_k) \tag{23}$$

In particular, $\tilde{b}$ is trivial on $\Omega(A)$ and

$$(b\Omega(A)) \subset \Omega(A)$$

$$(b \circ d + d \circ b)(\alpha_{n-1} \cdot da) = \alpha_{n-1} \cdot da - (-1)^{n-1} da \cdot \alpha_{n-1} =: [\alpha_{n-1}, da] \tag{24}$$

where $a \in A$, $da \in \Omega^1(A)$, $\alpha_{n-1} \in \Omega^{n-1}(A)$, and $\alpha_{n-1} \cdot da \in \Omega^n(A)$.

We also have the following proposition (Connes, 1986; Karoubi, 1982):

Proposition 3. The cohomology $H(\Omega(A))$ of $\Omega(A)$ is trivial, i.e., $H^0(\Omega(A)) = C$ and $H^n(\Omega(A)) = 0$ for $n \geq 0$.

Proof. From the proof of Proposition 1, Proposition 3 follows from the fact that one has

$$k_\omega(\Omega^{n+1}(A)) \subset \Omega^n(A) \tag{25}$$

for $n \geq 0$, so k_ω is also a contracting homotopy for the subcomplex

$$0 \xrightarrow{0} C \xrightarrow{i} \Omega^0(A) \xrightarrow{d} \Omega^1(A) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(A) \xrightarrow{d} \Omega^{n+1}(A) \xrightarrow{d} \cdots \quad \blacksquare \tag{26}$$

2.3. The Canonical Operation of Der(A) in $\Omega(A)$

We obtain the following proposition from the above results.

Proposition 4. For any $\chi \in \mathrm{Der}(A)$, the unique homomorphism of A-bimodules

$$i_\chi\colon \quad \Omega^1(A) \to A = \Omega^0(A)$$

such that $\chi = i_\chi \circ d$ [where $d\colon A \to \Omega^1(A)$ is a derivation of A of degree $+1$], extends uniquely as an antiderivation of $\Omega(A)$ also denoted by i_χ. Define

$$L_\chi = i_\chi \circ d + d \circ i_\chi \tag{27a}$$

Then, i_χ is an antiderivation of *degree* -1, L_χ a derivation of degree 0, and we have for any $\chi_1, \chi_2 \in \mathrm{Der}(A)$

$$i_{\chi_1} \circ i_{\chi_2} + i_{\chi_2} \circ i_{\chi_1} = 0 \tag{27b}$$

$$L_{\chi_1} \circ i_{\chi_2} - i_{\chi_2} \circ L_{\chi_1} = i_{[\chi_1, \chi_2]} \tag{27c}$$

$$L_{\chi_1} \circ L_{\chi_2} - L_{\chi_2} \circ L_{\chi_1} = L_{[\chi_1, \chi_2]} \tag{27d}$$

In the sense of Cartan (1950) [see also Greub *et al.* (1976) for the theory of operations], it becomes clear that we are in the presence of an operation of the Lie algebra Der(A) of derivations of A in the graded differential algebra $\Omega(A)$ of differential forms on A; it is called the *canonical operation* of Der(A) in $\Omega(A)$.

Notice that, for $\chi \in \text{Der}(A)$, L_χ is the restriction to $\Omega(A)$ of the canonical extension to $\tau(A)$ of χ.

2.4. Subalgebras of $\Omega(A)$

In the context of the theory of operations (Cartan, 1950; Greub *et al.*, 1976) and using the same terminology as in the principal fiber bundle theory, one has the following definitions.

2.4.1. The Horizontal Subalgebra $\Omega_H(A)$

An element α of $\Omega(A)$ is called *horizontal* if

$$i_\chi(\alpha) = 0 \tag{28}$$

for all $\chi \in \text{Der}(A)$. The set $\Omega_H(A)$ of all horizontal elements of $\Omega(A)$ is a graded subalgebra of $\Omega(A)$ stable by the derivations L_χ, $\chi \in \text{Der}(A)$. Since $\Omega^0(A) \equiv A$, then $\Omega_H(A)$ is also a subbimodule of $\Omega(A)$.

2.4.2. The Invariant Subalgebra $\Omega_I(A)$

An element α of $\Omega(A)$ is called *invariant* if

$$L_\chi(\alpha) = 0 \tag{29}$$

for all $\chi \in \text{Der}(A)$. The set $\Omega_I(A)$ of all invariant elements of $\Omega(A)$ is a graded differential subalgebra of $\Omega(A)$. The cohomology of $\Omega_I(A)$ is denoted by $H_I(\Omega(A))$ and is called the *invariant cohomology* of $\Omega(A)$.

2.4.3. The Basic Subalgebra $\Omega_B(A)$

An element χ of $\Omega(A)$ is called *basic* if it is both horizontal and invariant. The set $\Omega_B(A)$ of all basic elements of $\Omega(A)$ forms a graded differential subalgebra of $\Omega_I(A)$ and then also of $\Omega(A)$. The cohomology of $\Omega_B(A)$ is denoted by $H_B(\Omega(A))$ and is called the *basic cohomology* of $\Omega(A)$.

$\Omega_B(A)$ is also the set of all $\alpha \in \Omega(A)$ such that α and $d\alpha$ are horizontal. Remark that $\Omega_I(A)$ and $\Omega_B(A)$ are generally not A-bimodules, but they are A_I-bimodules, where A_I is the *invariant* subalgebra of A defined by

$$A_I = \{a \in A \,/\, \chi(a) = 0, \forall\chi \in \mathrm{Der}(A)\} = \Omega_I^0(A) \subset A \tag{30}$$

As mentioned in Section 1, one goal of this approach is to give a good generalization of the notion of differentiable structure, and in this context one already expects that

$$A_I = C\mathbb{1} \tag{31}$$

where $\mathbb{1}$ is the unit of A.

Finally, one has the following result.

Proposition 5. Assume that a linear form ω on A exists such that

$$\omega(\mathbb{1}) = 1$$

and

$$\omega \circ \chi = 0$$

for all $\chi \in \mathrm{Der}(A)$. Then, $H_I(\Omega(A))$ is trivial, i.e., $H_I^0(\Omega(A)) = C$ and $H_I^n(\Omega(A)) = 0$ for $n \geq 1$.

Proof. As a continuation of the proofs of Propositions 1 and 3, here the property $\omega \circ \chi = 0$ implies that k_ω leaves stable the invariant subcomplex

$$0 \to C \xrightarrow{i_0} \Omega_I^0(A) \xrightarrow{d} \Omega_I^1(A) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_I^n(A) \xrightarrow{d} \Omega_I^{n+1}(A) \xrightarrow{d} \cdots \tag{32}$$

where i_0 denotes the restriction of i: $C \to \Omega^0(A)$ to $\Omega_I^0(A)$. ■

2.5. The Graded Differential Algebra $\Omega_{\mathrm{Der}}(A)$

2.5.1. *The Complex $C^* = C(\mathrm{Der}(A); A)$*

Bearing always in mind that the Lie algebra Der(A) of derivations of A, which is a generalization of the Lie algebra $V(M)$ of vector fields on the manifold M, acts by derivations on A, we will now construct the complex $C^* = C(\mathrm{Der}(A); A)$ of A-valued cochains of Der(A). First, recall that a p-cochain α_p on the Lie algebra Der(A) with values in A is a p-linear antisymmetric mapping of Der(A) in A, i.e.,

$$\alpha_p\colon \quad [\mathrm{Der}(A)]^p \equiv \bigwedge^p(\mathrm{Der}(A)) \to A \tag{33}$$

Let $C^p(\mathrm{Der}(A); A)$ denote the space of p-cochains of Der(A) with values in A and $C^* = C(\mathrm{Der}(A); A)$ the graded vector space:

$$C^* = C(\mathrm{Der}(A); A) = \bigoplus_{p\in\mathbf{N}} C^p(\mathrm{Der}(A); A) \tag{34}$$

This space is a graded A-module naturally equipped with a homogeneous differential d called the *coboundary operator* such that

$$d: \quad C^p(\mathrm{Der}(A); A) \to C^{p+1}(\mathrm{Der}(A); A)$$

$$d\alpha_p(\chi_0, \ldots, \chi_p) = \sum_{0 \le k \le p} (-1)^k \chi_k \alpha_p(\chi_0, \ldots, \overset{k}{\vee}, \ldots, \chi_p) + \sum_{0 \le r \le s \le p} (-1)^{r+s} \alpha_p([\chi_r, \chi_s], \chi_0, \ldots, \overset{r}{\vee}, \ldots, \overset{s}{\vee}, \ldots \chi_p) \tag{35}$$

for $\chi_0, \chi_1, \ldots, \chi_p \in \mathrm{Der}(A)$ and $\overset{k}{\vee}$ means omission of χ_k (Greub *et al.*, 1976; Koszul, 1950; Chevalley and Eilenberg, 1948).

With this operator, obtained by using the product of A and antisymmetrization on the arguments in $\mathrm{Der}(A)$, the complex C^* becomes a graded differential algebra with A as the subalgebra $C^0(\mathrm{Der}(A); A)$ of elements of degree zero, and with d a nilpotent antiderivation of degree $+1$.

The kernel of d is the module $Z(C^*)$ of *cocycles* and its image is the module $B(C^*)$ of *coboundaries*. Then, $B(C^*) \subset Z(C^*)$ and the *cohomology module* $H(C^*)$ is defined to be the quotient $Z(C^*)/B(C^*)$.

2.5.2. *The Subalgebra* $\Omega_{\mathrm{Der}}(A)$

From the universal Property 2 of $\Omega(A)$, the identity map

$$\mathrm{Id}_A: \quad A \to C^0(\mathrm{Der}(A); A) \equiv A$$

lifts uniquely as a homomorphism of graded differential algebras:

$$\Phi: \quad \Omega(A) \to C(\mathrm{Der}(A); A) \tag{36}$$

Generally, this homomorphism is neither surjective nor injective. Its kernel will be described in Section 2.6 and its image, denoted by $\Omega_{\mathrm{Der}}(A)$, is defined to be the smallest graded differential subalgebra of C^* that contains A. The graded differential algebra $\Omega_{\mathrm{Der}}(A)$ is a quotient of $\Omega(A)$ and its elements of degree $n \in \mathbf{N}$ are finite sums of terms of the form

$$a_0 da_1 da_2 \cdots da_n \tag{37}$$

with $a_i \in A$ and d the differential of the complex C^*. If $A = A_0 = C^\infty(M)$, where M is a good smooth manifold (say a finite-dimensional connected paracompact C^∞-manifold), $\Omega_{\mathrm{Der}}(A)$ coincides with the differential algebra $\Omega(A)$ of differential forms on M.

In fact, $\mathrm{Der}(A)$ is nothing else than the Lie algebra of vector fields on M and $\Omega_{\mathrm{Der}}(A)$ may be considered as a natural *noncommutative* generalization of the algebra $\Omega(M)$ of differential forms.

The cohomology $H(\Omega_{\text{Der}}(A))$ of $\Omega_{\text{Der}}(A)$ will be denoted by $H_{\text{Der}}(A)$. It is also a graded algebra.

To treat some particular applications, one needs to use the notion of *completion* $\hat{\Omega}_{\text{Der}}(A)$ of $\Omega_{\text{Der}}(A)$. It is defined as being the set of elements $\alpha \in C^*$ such that for any finite-dimensional subspace F of Der(A), there is an element $\alpha_F \in \Omega_{\text{Der}}(A)$ such that

$$\alpha(\chi_1, \ldots, \chi_p) = \alpha_F(\chi_1, \ldots, \chi_p) \tag{38}$$

for $\chi_1, \ldots, \chi_p \in F$. Obviously, $\hat{\Omega}_{\text{Der}}(A)$ is also a graded differential subalgebra of C^*. It is also clear that, in the case of $A = A_0$ with M the above-chosen manifold, one has $\hat{\Omega}_{\text{Der}}(A_0) \equiv \Omega_{\text{Der}}(A_0)$. We shall not make a distinction between $\hat{\Omega}_{\text{Der}}(A)$ and $\Omega_{\text{Der}}(A)$ except in the case where they do not coincide. In fact, the only example that we shall meet where $\hat{\Omega}_{\text{Der}}(A)$ is bigger than $\Omega_{\text{Der}}(A)$ is the one of $A_\hbar$ (see Section 5.2.).

2.5.3. The Operation of Der(A) in $\Omega_{Der}(A)$

As in Section 2.3, we can also define an operation of Der(A) in C^*. Let $\chi, \chi_1, \chi_2, \ldots, \chi_{n-1} \in \text{Der}(A)$ and $\alpha_n \in C^n(\text{Der}(A); A)$. One may introduce an operation i_χ of Der(A) in the graded differential algebra C^* defined by $(n \geq 1)$

$$i_\chi: \quad C^n(\text{Der}(A); A) \to C^{n-1}(\text{Der}(A); A)$$

such that

$$i_\chi\alpha_n(\chi_1, \ldots, \chi_{n-1}) = \alpha_n(\chi, \chi_1, \ldots, \chi_{n-1}) \tag{39}$$

Then, i_χ is an antiderivation of degree -1 and we have for all $\chi \in \text{Der}(A)$

$$i_\chi \circ \Phi = \Phi \circ i_\chi \tag{40}$$

[where Φ is given by equation (36)], which implies that i_χ leaves stable $\Omega_{\text{Der}}(A)$.

Let us now define the derivation of degree zero L_χ by [see equation (27a)]

$$L_\chi = i_\chi \circ d + d \circ i_\chi \tag{41a}$$

and we have

$$[i_{\chi_1}, i_{\chi_2}]_+ = 0 \tag{41b}$$

$$[L_{\chi_1}, i_{\chi_2}]_- = i_{[\chi_1,\chi_2]} \tag{41c}$$

$$[L_{\chi_1}, L_{\chi_2}]_- = L_{[\chi_1,\chi_2]} \tag{41d}$$

So, we obtain an operation of the Lie algebra Der(A) of derivations of A in C^* in the sense of Cartan (1950).

The operation i_χ may be *restricted* to $\Omega_{\mathrm{Der}}(A)$ [and also to the completion $\hat{\Omega}_{\mathrm{Der}}(A)$] so that we have an operation of Der(A) in $\Omega_{\mathrm{Der}}(A)$:

$$i_\chi(\Omega_{\mathrm{Der}}(A)) \subset \Omega_{\mathrm{Der}}(A)$$

Effectively, if $\alpha_p \in \Omega^p_{\mathrm{Der}}(A)$ with $p \geq 1$, then

$$i_\chi(\alpha_p) = 0, \qquad \forall \chi \in \mathrm{Der}(A) \Rightarrow \alpha_p = 0 \tag{42}$$

It follows that the only *horizontal* elements of $\Omega_{\mathrm{Der}}(A) = \oplus_{p \in \mathbf{N}} \Omega^p_{\mathrm{Der}}(A)$ are the elements of $A = \Omega^0_{\mathrm{Der}}(A)$, and that, in this sense, $\Omega_{\mathrm{Der}}(A)$ is a *restriction* of $\Omega(A)$ [it is a quotient of $\Omega(A)$]. Then, we may define an *operation* of Der(A) in $\Omega_{\mathrm{Der}}(A)$. In this case, we have

$$L_\chi(a) = \chi(a) \tag{43}$$

for $a \in A = \Omega^0_{\mathrm{Der}}(A)$.

An element $\alpha \in \Omega_{\mathrm{Der}}(A)$ will be called *invariant* if [see equation (29)]

$$L_\chi(\alpha) = 0$$

for any $\chi \in \mathrm{Der}(A)$.

In fact, here i_χ and L_χ generalize the usual notions of inner product and Lie derivative, respectively.

Finally, if we consider A to be a $C^\star$-algebra, we may define an antilinear *involution* on Der(A) and its extension to an antilinear involution on $\Omega_{\mathrm{Der}}(A)$ by setting

$$\chi^\star(a) = (\chi(a^\star))^\star \tag{44}$$

for $\chi \in \mathrm{Der}(A)$ and $a \in A$, and

$$\alpha_p^\star(\chi_1, \ldots, \chi_p) = (\alpha_p(\chi_1^\star, \ldots, \chi_p^\star))^\star \tag{45}$$

for $\alpha_p \in \Omega^p_{\mathrm{Der}}(A)$ and $\chi_i \in \mathrm{Der}(A)$, respectively.

Then, $\Omega_{\mathrm{Der}}(A)$ becomes a *graded differential* $C^\star$-algebra:

$$d(\alpha^\star) = (d\alpha)^\star \tag{46}$$

for $\alpha \in \Omega_{\mathrm{Der}}(A)$, and

$$(\alpha_p \wedge \beta_q)^\star = (-1)^{pq} \beta_q^\star \wedge \alpha_p^\star \tag{47}$$

where $\alpha_p \in \Omega^p_{\mathrm{Der}}(A)$ and $\beta_q \in \Omega^q_{\mathrm{Der}}(A)$.

The derivations $\chi \in \mathrm{Der}(A)$ and the elements $\alpha \in \Omega_{\mathrm{Der}}(A)$ are called *real* if they satisfy

$$\chi = \chi^\star \tag{48}$$

and

$$\alpha = \alpha^\star \tag{49}$$

respectively.

2.6. The Filtration of $\Omega(A)$

Although $\Omega(A)$ is not *graded-commutative*, a filtration of $\Omega(A)$ associated with the above-defined operation of Der(A) may be defined as in the usual graded-commutative case (Greub *et al.*, 1976; Koszul, 1950).

Namely, one defines subspaces $F^p(\Omega^n(A)) \subset \Omega^n(A)$, $p \leq n$, by

$$\begin{aligned} F^p(\Omega^n(A)) &= \{\alpha_n \in \Omega^n(A) \,/\, i_{\chi_1} \cdots i_{\chi_{n-p+1}}(\alpha) \\ &= 0, \forall \chi_1, \ldots, \chi_{n-p+1} \in \mathrm{Der}(A)\} \end{aligned} \tag{50}$$

and sets

$$F^p(\Omega(A)) = \bigoplus_{n \geq p} F^p(\Omega^n(A)) \tag{51a}$$

with

$$F^0(\Omega(A)) \equiv \Omega(A) \tag{51b}$$

$$F^{p+1}(\Omega(A)) \subset F^p(\Omega(A)) \tag{51c}$$

$$F^p(\Omega(A)) \otimes F^q(\Omega(A)) \subset F^{p+q}(\Omega(A)) \tag{51d}$$

$$dF^p(\Omega(A)) \subset F^p(\Omega(A)) \tag{51e}$$

Then, in particular, the $F^p(\Omega(A))$ are graded two-sided ideals of $\Omega(A)$ and $F^{p+1}(\Omega(A))$ is a two-sided ideal in $F^p(\Omega(A))$.

Furthermore, the spaces $F^p(\Omega(A))$ are stable by the operations i_χ and L_χ for $\chi \in \mathrm{Der}(A)$ [in view of equation (27c)] and by the differential d [in view of equation (27a)]. This implies, in particular, that the $F^p(\Omega(A))$ define a *filtration* of graded differential algebra on $\Omega(A)$. This filtration is called the *natural filtration* or *first filtration* of $\Omega(A)$.

To such a filtration corresponds a convergent spectral sequence (E_k, d_k), $k \in \mathbf{N}$, where E_k is a bigraded algebra:

$$E_k = \bigoplus_{p,q \in \mathbf{N}} E_k^{p,q} \tag{52a}$$

with a differential d_k *homogeneous of bidegree* $(k, 1 - k)$, and with

$$E_0 = \bigoplus_{n \in \mathbf{N}} (F^n(\Omega(A))/F^{n+1}(\Omega(A))) \tag{52b}$$

In view of the triviality of the cohomology $H(\Omega(A))$ (see Proposition 3), E_∞ is the trivial bigraded algebra with

$$E_\infty^{0,0} = C \tag{53a}$$

$$E_\infty^{p,q} = 0 \qquad \text{for} \quad p + q \geq 1 \tag{53b}$$

Consider the bigraded space

$$F = \bigoplus_{p,q \in \mathbf{N}} F^{p,q} \tag{54a}$$

where

$$F^{p,q} = F^p(\Omega^{p+q}(A)) \tag{54b}$$

Then, one has

$$dF^{p,q} \subset F^{p,q+1} \tag{54a}$$

$$F^{p,q} \otimes F^{r,s} \subset F^{p+r,q+s} \tag{54b}$$

So, F has the canonical structure of a bigraded differential algebra with differential d_0 homogeneous of bidegree (0, 1) induced by d. In this case, $\Omega(A)$ coincides with the graded differential subalgebra:

$$F^{0,*} = \bigoplus_{n \in \mathbf{N}} F^{0,n} = \bigoplus_{n \in \mathbf{N}} F^0(\Omega^n(A)) = F^0(\Omega(A)) \equiv \Omega(A) \tag{55}$$

This structure defines on the above-mentioned associated graded space E_0 a structure of *bigraded algebra* with

$$E_0^{p,q} = F^{p,q}/F^{p+1,q-1} \tag{56a}$$

and a structure of *differential algebra* with differential d_0 such that

$$dE_0^{p,q} \subset E_0^{p,q+1} \tag{56b}$$

It is obvious that the kernel of the homomorphism Φ: $\Omega(A) \rightarrow C^*$ introduced below [see equation (36)] is, by definition, $F^1(\Omega(A))$. So, one has

$$E_0^{0,p} = \Omega_{\mathrm{Der}}^p(A) \tag{56c}$$

$$E_0^{0,*} = \bigoplus_{p \in \mathbf{N}} E_0^{0,p} = \Omega_{\mathrm{Der}}(A) \tag{56d}$$

On the other hand, $\chi \rightarrow L_\chi$ defines an *action* of Der(A) by derivations of degree zero on the graded algebra $\Omega_H(A)$. The complex $C_H^* = C(\mathrm{Der}(A); \Omega_H(A))$ of cochains on Der(A) with values in $\Omega_H(A)$ is a bigraded algebra with differential d_0 such that

$$C_H^* = \bigoplus_{p,q \in \mathbf{N}} C_H^{p,q} \tag{57a}$$

$$C_H^{p,q} = C^q(\mathrm{Der}(A); \Omega_H^p(A)) \tag{57b}$$

$$d_0 C_H^{p,q} \subset C_H^{p,q+1} \tag{57c}$$

and

$$C^* = C(\mathrm{Der}(A); A) = C_H^{0,*} = \bigoplus_{n \in \mathbf{N}} C_H^{0,n} \tag{57d}$$

Now, deduce from the homomorphism Φ a homomorphism

$$\Phi' : \quad F \to C_H^* \tag{58a}$$

with

$$\Phi_p' : \quad F^p(\Omega(A) \to C(\mathrm{Der}(A); \Omega_H^p(A)) \tag{58b}$$

Let α be an element of $F^{p,q} = F^p(\Omega^{p+q}(A))$. Then, by definition, one has

$$i_\chi i_{\chi_1} \cdots i_{\chi_q}(\alpha) = 0 \tag{59}$$

for any $\chi, \chi_1, \ldots, \chi_q \in \mathrm{Der}(A)$. This means that

$$(\chi_1, \chi_2, \ldots, \chi_q) \to i_{\chi_1} i_{\chi_2} \cdots i_{\chi_q}(\alpha) \tag{60}$$

is an element of $\Phi'(\alpha) \in C_H^{p,q}$.

The application Φ' so defined is clearly a homomorphism of bigraded algebras and differential algebras from F into C_H^*.

The kernel of the homomorphism

$$\Phi_{p,q}' : \quad F^{p,q} \to C_H^{p,q} \tag{61}$$

is manifestly $F^{p+1,q-1} \subset F^{p,q}$, so the image of Φ' is canonically $\Phi'(F) = E_0 \subset C_H^*$ as differential algebra and bigraded algebra.

In particular, one has

$$E_0^{n,0} = \Omega_H^n(A) \tag{62a}$$

$$E_0^{0,n} = \Omega_{\mathrm{Der}}^n(A) \tag{62b}$$

From the isomorphism

$$E_1 \sim H(E_0, d_0) \tag{63a}$$

one has

$$E_1^{n,0} = \Omega_B^n(A) \tag{63b}$$

$$E_1^{0,n} = H_{\mathrm{Der}}^n(A) \tag{63c}$$

It is easy to prove that d_1 induces on $E_1^{*,0} = \Omega_B(A)$ the differential d. Hence, one has

$$E_2^{n,0} = H_B^n(\Omega(A)) \tag{64}$$

and then we see that the spectral sequence ties between the cohomology algebras $H_{\mathrm{Der}}(A)$ and $H_B(\Omega(A))$.

2.7. The Operation of Int(A) in $\Omega_{\mathrm{Der}}(A)$

Let Int(A) be the Lie algebra of inner derivations of A. It is a Lie subalgebra (and also an ideal) of Der(A). One may define an operation of the Lie algebra Int(A) in the graded differential algebra $\Omega_{\mathrm{Der}}(A)$ by restriction to Int(A) of the operation of Der(A) in $\Omega_{\mathrm{Der}}(A)$ presented in Section 2.5.3.

The set of *basic* elements of $\Omega_{\mathrm{Der}}(A)$ for the operation of Int(A) is a graded differential subalgebra $\Omega_{\mathrm{Out}}(A)$ of $\Omega_{\mathrm{Der}}(A)$ defined by[3]

$$\Omega_{\mathrm{Out}}(A) = \{\alpha \in \Omega_{\mathrm{Der}}(A) \,/\, i_\chi(\alpha) = 0 \text{ and } L_\chi(\alpha) = 0, \forall \chi \in \mathrm{Int}(A)\} \tag{65}$$

On the other hand, one has a canonical homomorphism

$$\psi: \quad C(\mathrm{Der}(A); A) \to C(\mathrm{Int}(A); A) \tag{66}$$

The image $\psi(\Omega_{\mathrm{Der}}(A))$ of $\Omega_{\mathrm{Der}}(A)$ by this homomorphism is a graded differential subalgebra $\Omega_{\mathrm{Int}}(A)$ of $C(\mathrm{Int}(A); A)$.

2.8. The Filtration of $\Omega_{\mathrm{Der}}(A)$

As in Section 2.6, one may define the filtration of $\Omega_{\mathrm{Der}}(A)$ associated with the operation of Int(A) in $\Omega_{\mathrm{Der}}(A)$. In this case, the corresponding spectral sequence $(\tilde{E}_k, \tilde{d}_k)$, $k \in \mathbf{N}$, ties the cohomologies $H_{\mathrm{Out}}(A)$, $H_{\mathrm{Int}}(A)$, and $H_{\mathrm{Der}}(A)$ of $\Omega_{\mathrm{Out}}(A)$, $\Omega_{\mathrm{Int}}(A)$, and $\Omega_{\mathrm{Der}}(A)$, respectively. In particular, one has

$$\tilde{E}_1^{p,0} = \Omega_{\mathrm{Out}}^p(A) \tag{67a}$$

$$\tilde{E}_1^{0,q} = H_{\mathrm{Int}}^q(A) \tag{67b}$$

$$\tilde{E}_2^{p,0} = H_{\mathrm{Out}}^p(A) \tag{67c}$$

and $\tilde{E}_\infty$ is the graded space associated with the cohomology $H_{\mathrm{Der}}(A)$ for the induced filtration (Koszul, 1950).

[3] It is important to point out that Ω_{Out} is a functor of associative algebras and that Ω_{Out}, H_{Out} and $H^1(A,A) = \mathrm{Out}(A)$ are Morita invariants. This leads to the construction of a projective A-bimodule of finite rank and the study of the cyclic cohomology of the underlying manifold with a nontrivial cocycle. We plan to treat this question in a future paper.

3. EXAMPLES

3.1. The Limit Case $A = A_0 = C^\infty(M)$

This case is the most trivial one, since all the notions introduced above to define a *noncommutative* differential calculus reduce to the classical notions of the usual *commutative* differential geometry.

In fact, for $A \equiv A_0 = C^\infty(M)$, where M is a finite-dimensional paracompact connected C^∞-manifold, Der(A) reduces to the Lie algebra $V(M)$ of vector fields ∂_k on M and $\Omega^n(A)$ to the subspace of $C^\infty(M)^{\otimes n+1}$ of functions on M^{n+1} that cancel out when two consecutive arguments coincide.

One has

$$\Omega_I^0(A) \equiv \Omega_B^0(A) \equiv C \tag{68a}$$

i.e., the set of constant functions on M, and

$$\Omega_I^n(A) \equiv \Omega_B^n(A) = 0 \tag{68b}$$

for $n \geq 1$.

Hence, one obtains *trivial* invariant and basic cohomologies of $\Omega(A)$:

$$H_I^0(\Omega(A)) = H_B^0(\Omega(A)) = C$$
$$H_I^n(\Omega(A)) = H_B^n(\Omega(A)) = 0 \qquad \text{for} \quad n \geq 1 \tag{69}$$

On the other hand, it is clear that, by definition, $\Omega_{\text{Der}}(A)$ represents the graded differential algebra $\Omega(M)$ of differential forms on M. The induced cohomology $H_{\text{Der}}(A) = H(\Omega_{\text{Der}}(A))$ is nothing else than the De Rham cohomology $H_{\text{DR}}(M)$ of the manifold M.

Finally, for the case $A \equiv A_0$, one has

$$\Omega_{\text{Out}}(A) \equiv \Omega_{\text{Der}}(A) \equiv \Omega(M) \tag{70a}$$

$$H_{\text{Out}}(A) \equiv H_{\text{Der}}(A) \equiv H_{\text{DR}}(M) \tag{70b}$$

and

$$\Omega_{\text{Int}}^n(A) = 0 \qquad \text{for} \quad n \geq 1 \tag{70c}$$

3.2. The Case $A = \mathbf{M}_n(C)$

3.2.1. Differential Calculus

In this case (Dubois-Violette *et al.*, 1990a), where $\mathbf{M}_n(C)$ represents the algebra End(C) of endomorphisms of C^n (i.e., the set of complex $n \times n$ matrices, $n \geq 2$), the complex Lie algebra Der(A) reduces to $sl(n, C)$, since all the derivations of $A = \mathbf{M}_n(C)$ are *inner*. It follows that the complex (real) Lie algebra $\text{Der}(\mathbf{M}_n(C))$ $[\text{Der}_{\mathbf{R}}(\mathbf{M}_n(C))]$ reduces to the Lie algebra $sl(n)[su(n)]$.

A derivation χ of $\mathbf{M}_n(C)$ is called *real* in the sense that it preserves Hermiticity, i.e., for $E \in \mathbf{M}_n(C)$ one has [see equations (44) and (48)]

$$\chi(E) = \chi^\star(E) = (\chi(E^\star))^\star \tag{71}$$

Here, one may choose

$$\omega = \frac{1}{n}\,\mathrm{tr}(\cdot) \tag{72}$$

as an example of a linear form on A (i.e., $\omega \in A^\star$) that satisfies the conditions of Propositions 1, 3, and 5. This choice implies that the *invariant* cohomology $H_I(\Omega(A))$ of $\Omega(A)$ is *trivial.*

On the other hand, it is easy to see that the homomorphism Φ [see equation (36)] induces an injective application of $\Omega^1(\mathbf{M}_n(C))$ in $C^1(sl(n); \mathbf{M}_n(C))$. Since

$$\dim_C \Omega^1(\mathbf{M}_n(C)) = \dim_C C^1(sl(n); \mathbf{M}_n(C)) = n^2(n^2 - 1)$$

one has

$$\Omega^1(\mathbf{M}_n(C)) \equiv C^1(sl(n); \mathbf{M}_n(C)) \tag{73}$$

Furthermore, it is shown that the smallest differential subalgebra $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ of $C(\mathrm{Der}(\mathbf{M}_n(C)); \mathbf{M}_n(C))$ which contains $\mathbf{M}_n(C)$ is $C(\mathrm{Der}(\mathbf{M}_n(C)); \mathbf{M}_n(C))$ itself, i.e.,

$$\Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) \equiv C(sl(n, C); \mathbf{M}_n(C)) \equiv \Lambda\, sl(n, C)^* \otimes \mathbf{M}_n(C) \tag{74}$$

Any element α_p of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ is a p-linear antisymmetric mapping

$$\alpha_p\colon \quad (\mathrm{Der}(\mathbf{M}_n(C))^p \to \mathbf{M}_n(C)$$

$$(\chi_1, \ldots, \chi_p) \to \alpha_p(\chi_1, \ldots, \chi_p)$$

and its differential $d\alpha_p \in \Omega^{p+1}_{\mathrm{Der}}(\mathbf{M}_n(C))$ is given by equation (35).

The only elements of $\mathbf{M}_n(C)$ that are *invariant* under $\mathrm{Der}(\mathbf{M}_n(C))$ [i.e., by the adjoint action of $sl(n, C)$] are the multiples of $\mathbb{1} \in \mathbf{M}_n(C)$. Thus, it follows from the semisimplicity of $sl(n, C)$ that the cohomology $H_{\mathrm{Der}}(\mathbf{M}_n(C))$ of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ reduces to the Lie algebra cohomology $H^*(sl(n, C))$:

$$H_{\mathrm{Der}}(M_n(C)) \equiv H^*(sl(n, C)) \tag{75}$$

This cohomology is well known. It is the free graded-commutative algebra with unit $\Lambda(\alpha_3, \ldots, \alpha_{2n-1})$ generated by α_{2p-1}, $p = 2, 3, \ldots, n$, with α_{2p-1} of degree $2p - 1$. In particular, one has

$$H^1_{\mathrm{Der}}(\mathbf{M}_n(C)) = H^2_{\mathrm{Der}}(\mathbf{M}_n(C)) = 0 \tag{76a}$$

such that every closed element of $\Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C))$ [or $\Omega^2_{\mathrm{Der}}(M_n(C))$] is *exact*, and also

$$H^0_{\text{Der}}(\mathbf{M}_n(C)) = H^3_{\text{Der}}(\mathbf{M}_n(C)) = H^{n^2-1}_{\text{Der}}(\mathbf{M}_n(C)) = C \tag{76b}$$

Hence, the cohomology $H_{\text{Der}}(\mathbf{M}_n(C))$ depends on the integer n. This is tied to the fact that, in the case $A = \mathbf{M}_n(C)$, one is only concerned with inner derivations.

As in Section 2.5.3, there is an operation of the Lie algebra $\text{Der}(\mathbf{M}_n(C))$ in the graded differential algebra $\Omega_{\text{Der}}(\mathbf{M}_n(C))$ defined as follows:

For any $\chi \in \text{Der}(\mathbf{M}_n(C))$, one defines an antiderivation of degree -1 of $\Omega_{\text{Der}}(\mathbf{M}_n(C))$ by equation (39) and

$$i_\chi(\Omega^0_{\text{Der}}(\mathbf{M}_n(C))) = 0 \tag{77}$$

Then, L_χ [see equation (41a)] is a derivation of degree zero of $\Omega_{\text{Der}}(\mathbf{M}_n(C))$ which extends χ. Here, i_χ (resp. L_χ) is the analog of the inner product of forms by a vector field (resp. of the Lie derivative of forms by a vector field) and one has the characteristic relations (41b–41d).

An element α of $\Omega_{\text{Der}}(\mathbf{M}_n(C))$ is called *invariant* if $L_\chi(\alpha) = 0$ for any $\chi \in \text{Der}(\mathbf{M}_n(C))$. The set of the invariant elements forms a graded differential subalgebra with unit of $\Omega_{\text{Der}}(\mathbf{M}_n(C))$.

Moreover, one has

$$\Omega_{\text{Der}}(\mathbf{M}_n(C)) \equiv \Omega_{\text{Int}}(M_n(C)) \tag{78a}$$

and

$$\Omega^0_{\text{Out}}(\mathbf{M}_n(C)) = C\mathbb{1} \tag{78b}$$

$$\Omega^n_{\text{Out}}(\mathbf{M}_n(C)) = 0 \qquad \text{for} \quad n \geq 1 \tag{78c}$$

3.2.2. Presentation of $\Omega_{Der}(\mathbf{M}_n(C))$ Associated with a Basis

Let $\{E_k\}$, $k \in I = \{1, 2, \ldots, n^2-1\}$, be a basis of self-adjoint traceless $n \times n$ matrices. Then, $\{\mathbb{1}, E_k\}$ is a basis of $\mathbf{M}_n(C)$ consisting of Hermitian matrices.

As noted in Section 3.2.1, the derivations of $\mathbf{M}_n(C)$ are all inner, so the complex Lie algebra $\text{Der}(\mathbf{M}_n(C))$ reduces to $sl(n)$ and the real Lie algebra $\text{Der}_{\mathbf{R}}(\mathbf{M}_n(C))$ reduces to $su(n)$. We will restrict ourselves to the real case.

One has the following multiplication table:

$$E_k \cdot E_l = K_{kl}\mathbb{1} + \left(S^m_{kl} - \frac{i}{2}\, C^m_{kl}\right) E_m \tag{79}$$

where K_{kl} are the components of the Killing form of $su(n)$ given by

$$K_{kl} = K_{lk} = \frac{1}{n}\,\text{tr}(E_k \cdot E_l) \tag{80}$$

(in the case where $K_{kl} = \delta_{kl}$, the basis becomes orthonormal), and

$$S_{kl}^{m} = S_{lk}^{m} \tag{81}$$

and

$$C_{kl}^{m} = -C_{lk}^{m} \tag{82}$$

are real numbers. The latter correspond canonically to structure constants of $su(n)$ [i.e., to the components of three ad-invariant $su(n)$ tensors]. Then, the generators $\mathrm{i}E_k$ span the Lie algebra $su(n)$:

$$[\mathrm{i}E_k, \mathrm{i}E_l] = C_{kl}^{m}(\mathrm{i}E_m) \tag{83a}$$

and the Jacobi identity gives the following relation:

$$C_{ij}^{m} \cdot C_{km}^{n} + C_{jk}^{m} \cdot C_{im}^{n} + C_{ki}^{m} \cdot C_{jm}^{n} = 0 \tag{83b}$$

Thus,

$$S_{kl}^{l} = C_{kl}^{l} = 0 \tag{84}$$

and K_{kl}, $S_{km}^{p} \cdot S_{lp}^{m}$, and $C_{km}^{p} \cdot C_{lp}^{m}$ are the components of three bilinear forms proportional to the Killing form on $su(n)$, $(k, l, m, p \in I)$.

Finally, from (79) and (80) and using associativity, we obtain

$$\left(S_{kl}^{p} - \frac{\mathrm{i}}{2} C_{kl}^{p}\right)K_{mp} = \left(S_{lm}^{p} - \frac{\mathrm{i}}{2} C_{lm}^{p}\right)K_{kp} = \frac{1}{n} \operatorname{tr}(E_k \cdot E_l \cdot E_m) \tag{85}$$

and it follows that

$$S_{klm} = S_{kl}^{p}K_{pm} \tag{86a}$$

is completely symmetric and

$$C_{klm} = C_{kl}^{p}K_{pm} \tag{86b}$$

is completely antisymmetric. The quantities S_{klm} and C_{klm} satisfy also some other relations (see, for instance, Macfarlane *et al.*, 1968). From (86a) and (86b) the role of the components K_{kl} naturally appears as components of the Cartan–Killing metric tensor of $su(n)$ that lower or raise indices. We denote the components of the inverse matrix of (K_{kl}) by K^{kl} such that

$$K_{kl} \cdot K^{lm} = \delta_{k}^{m} \tag{87}$$

It follows from the above relations that

$$C_{kr}^{s} \cdot C_{ls}^{r} = -2n^2 K_{kl} \tag{88}$$

Recall that we are no longer interested in the notion of the manifold itself [in our case, there are only two maximal left (or right) ideals that eventually could be identified as points], but only in the algebra of *functions*. Our *functions* are therefore only 1, E_k, and their C-linear combinations.

It is quite straightforward to prove that the basis of all real derivations of $\mathbf{M}_n(C)$ is formed by the adjoint action of the generators of $su(n)$. Defining

$$e_k =: \mathrm{ad}(\mathrm{i}E_k) \tag{89}$$

we get a basis $\{e_k\}$, $k \in I$, of $\mathrm{Der}_{\mathbf{R}}(\mathbf{M}_n(C)) = su(n)$ such that

$$[e_k, e_l] = C_{kl}^m\, e_m \tag{90}$$

There are only $n^2 - 1$ independent *vector fields*. Contrary to ordinary differential geometry, these vector fields do not form a left module of $\mathbf{M}_n(C)$. This means that the linear operation defined by

$$(E_k e_l)(E_m) =: E_k \cdot e_l(E_m) \tag{91}$$

is not a derivation of $\mathbf{M}_n(C)$ since the Leibnitz rule is not satisfied. Only e_k satisfies this rule with respect to the associative multiplication in $\mathbf{M}_n(C)$:

$$e_k(E_l \cdot E_m) = e_k(E_l) \cdot E_m + E_l \cdot e_k(E_m) \tag{92}$$

Let us now construct the graded vector space $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) \equiv C(\mathrm{Der}(\mathbf{M}_n(C); \mathbf{M}_n(C))$ of exterior forms α_p defined as the linear mappings [see equation (33)]

$$\alpha_p\colon \quad \overset{p}{\bigwedge}(\mathrm{Der}(\mathbf{M}_n(C)) \to \mathbf{M}_n(C)$$

The basis $\{\theta^k\}$, $k \in I$, of 1-forms *dual* to the basis of real derivations $\{e_m\}$, $m \in I$, is therefore defined by

$$\theta^k(e_m) = \delta_m^k 1 \tag{93}$$

such that it is identified with $1 \otimes \overset{1}{\Lambda} sl(n, C)^* \subset \Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C))$.

By definition, the space of 1-forms is a module over $\mathbf{M}_n(C)$, i.e., one may also define the forms

$$E\theta^k = \theta^k E$$

$\forall E \in \mathbf{M}_n(C)$, and in particular

$$E_m\theta^k = \theta^k E_m \tag{94}$$

by taking their value on any e_p:

$$(E_m\theta^k)(e_p) = E_m(\theta^k(e_p)) = \delta_p^k E_m$$

The Grassmannian structure is introduced as usual, by defining the exterior product on the basis $\{\theta^k\}$:

$$(\theta^k \wedge \theta^m)(e_p, e_q) = \frac{1}{2}(\theta^k(e_p)\cdot\theta^m(e_q) - \theta^m(e_p)\cdot\theta^k(e_q))$$
$$= -(\theta^m \wedge \theta^k)(e_p, e_q) \quad (95)$$

The differential d of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ is defined independently of basis. First, on the 0-forms f (functions), one gets

$$df(\chi) = \chi(f)$$

with χ a vector field. In our case, this yields

$$d\mathbb{1} \equiv 0$$
$$dE_k(e_j) = e_j(E_k) = \mathrm{i}[E_j, E_k] = C^m_{jk}E_m \quad (96a)$$

which means that

$$dE_k = -C^m_{kp}E_m\theta^p \quad (96b)$$

In the general case, one could have chosen as a basis of 1-forms the set $\{dE_k\}$, $k \in I$, but the latter present a problem due to the noncommutativity

$$E_m\cdot dE_k \neq dE_k\cdot E_m \quad (97)$$

From now on, we will use the suitable basis $\{\theta^k\}$ for $\Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C))$. The formula (96b) can be inverted to yield

$$\theta^k = -\frac{\mathrm{i}}{n^2}K^{pq}K^{kr}E_p\cdot E_r dE_q \quad (98)$$

If one takes, by definition,

$$d(\theta^k \wedge \theta^m) = d\theta^k \wedge \theta^m - \theta^k \wedge d\theta^m \quad (99)$$

then

$$d^2 \equiv 0 \quad (100)$$

on any exterior product of p-forms. This result follows also from the Jacobi identity. Using the above relations, one obtains the important identity

$$d\theta^k = -\frac{1}{2}C^k_{pq}\theta^p \wedge \theta^q \quad (101)$$

which is the analog of the Maurer–Cartan identity on the group manifolds.

The relations (79), (94), (95), (96b), and (101) for generators E_k, θ^m and differential d give a presentation of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$.

Finally, since $\mathbf{M}_n(C)$ is the analog of complex functions and the real subspace of Hermitian matrices is the analog of real functions, then the analogs of real vector fields are the real derivations in the sense of equation (71) and the 1-forms θ^k must be considered as real. Therefore, one is led to define an antilinear involutive mapping, for $p \in I$,

$$\Omega^p_{\mathrm{Der}}(\mathbf{M}_n(C)) \to \Omega^p_{\mathrm{Der}}(\mathbf{M}_n(C))$$

such that

$$\alpha_p = a_{i_1 \cdots i_p}\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_p} \to \alpha_p^\star = a^\star_{i_1 \cdots i_p}\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_p} \tag{102}$$

The elements α_p of $\Omega^p_{\mathrm{Der}}(\mathbf{M}_n(C))$ satisfying

$$\alpha_p^\star = \alpha_p \Leftrightarrow a^\star_{i_1 \cdots i_p} = a_{i_1 \cdots i_p} \tag{103}$$

are said to be *real*. Since $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ is the analog of complex differential forms, the real vector space of real elements of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ is the analog of the space of real differential forms.

Notice that the mapping (102) is defined without any reference to the choice of the basis $\{E_k\}$, $k \in I$, of Hermitian matrices. Furthermore, if α_p is real, then $d\alpha_p$ is also real [see equation (47)]:

$$d(\alpha_p^\star) = (d\alpha_p)^\star$$

An element θ of $\Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C))$:

$$\theta = E_k\theta^k$$

is real [in view of equation (103)] and independent of the choice of the E_k. In fact, one has

$$\theta(\mathrm{ad}(\mathrm{i}E)) = E - \frac{1}{n}\,\mathrm{tr}(E) \tag{104}$$

where $E \in \mathbf{M}_n(C)$ (Dubois-Violette *et al.*, 1989a,b). Furthermore, θ is *invariant*:

$$L_X(\theta) = 0 \tag{105}$$

and any invariant element of $\Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C))$ is a scalar multiple of θ. This latter is called the *canonical invariant element* of $\Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C))$.

Finally, using equation (104), we can rewrite (96b) and (101) in the form

$$dE = \mathrm{i}[\theta, E], \qquad \forall E \in \mathbf{M}_n(C) \tag{106a}$$

and,

$$d(-\mathrm{i}\theta) + (-\mathrm{i}\theta)^2 = 0 \tag{106b}$$

respectively.

3.2.3. Integration

A notion of volume element and of integration of p-forms on p-chains can be introduced here, too. As left (or right) $\mathbf{M}_n(C)$-module, $\Omega_{\mathrm{Der}}^{n^2-1}(\mathbf{M}_n(C))$ is spanned by the unique generator $\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^{n^2-1}$. However, this element depends on the choice of the basis $\{E_k\}$. Let

$$k = \det(K_{kl}) \tag{107}$$

be the determinant of the real, positive-definite $(n^2 - 1) \times (n^2 - 1)$ matrix defined by equation (80). Then, the element

$$(|k|)^{1/2}\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^{n^2-1} \tag{108a}$$

depends only on the choice of the orientation of the basis $\{E_k\}$, $k \in I$. Thus, this real element is intrinsically defined up to a factor ± 1 fixed by the choice of an orientation. An arbitrary element $\alpha \in \Omega_{\mathrm{Der}}^{n^2-1}(\mathbf{M}_n(C))$ can be written as

$$\alpha = E(|k|)^{1/2}\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^{n^2-1} \tag{108b}$$

for $E \in \mathbf{M}_n(C)$.

One defines an *integral* over the total volume by means of a linear mapping

$$\int : \quad \Omega_{\mathrm{Der}}^{n^2-1}(\mathbf{M}_n(C)) \to C \tag{109a}$$

such that

$$\int \alpha = \frac{1}{n}\,\mathrm{tr}(E) \tag{109b}$$

One also has the following result.

Lemma 1. (a) The linear mapping $\int$ is a closed graded trace, i.e., in addition to (109b), one has

$$\int d\alpha = 0$$

$\forall \alpha \in \Omega_{\mathrm{Der}}^{n^2-1}(\mathbf{M}_n(C))$ and

$$\int \alpha_p \wedge \beta_q = (-1)^{pq} \int \beta_q \wedge \alpha_p$$

where $\alpha_p \in \Omega^p_{\mathrm{Der}}(\mathbf{M}_n(C))$, $\beta_q \in \Omega^q_{\mathrm{Der}}(\mathbf{M}_n(C))$, and $p + q = n^2 - 1$;

(b) $(|k|)^{1/2}\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^{n^2-1}$ is *invariant*, i.e.,

$$L_\chi((|k|)^{1/2}\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^{n^2-1}) = 0$$

$\forall\chi \in \mathrm{Der}(\mathbf{M}_n(C))$.

Proof. Statement (a) means that $(\Omega_{\mathrm{Der}}(\mathbf{M}_n(C)), \int)$ is a cycle of dimension $n^2 - 1$, in the sense of Connes (1986). It follows from (101) and from the complete antisymmetry of C_{klm} [see equation (86b)] that one has

$$d(\theta^{i_1} \wedge \theta^{i_2} \wedge \cdots \wedge \theta^{i_{n^2-1}}) = 0 \tag{110}$$

This implies that the element (108a) is invariant [i.e., statement (b) of the lemma] and also shows that the only contributions to $d\alpha$, α being an $(n^2 - 1)$-form, come from the differential of linear combinations of terms of the form

$$E_k\theta^{i_1} \wedge \theta^{i^2} \wedge \cdots \wedge \theta^{i_{n^2-1}}$$

Then, using equation (96b), one deduces that $d\alpha$ must be of the form

$$d\alpha = E\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^{n^2-1}$$

for some traceless matrix E. Therefore, it follows from equation (109b) that

$$\int d\alpha = 0$$

Finally, it is easy to check that

$$\int \alpha_p \wedge \beta_q = (-1)^{pq} \int \beta_q \wedge \alpha_p$$

by using equation (95). ■

3.2.4. Canonical Riemannian Structure

The form of the volume element (108a) looks like the volume element of a metric. This suggests the introduction of a flat metric defined by the symmetric 2-form

$$K = K_{kl}\theta^k \wedge \theta^p \tag{111}$$

belonging to

$$\mathbf{M}_n(C) \otimes \overset{2}{\Lambda} sl(n, C)^* \subset \Omega_{\mathrm{Der}}{}^1(\mathbf{M}_n(C)) \underset{\mathbf{M}_n(C)}{\otimes} \Omega_{\mathrm{Der}}{}^1(\mathbf{M}_n(C))$$

with K_{kl} being the Cartan–Killing metric of $su(n)$ [or $sl(n, C)$] given by equation (80) and its inverse is denoted by K^{kl} [equation (87)]. K is really the analog of an invariant Riemannian metric [for $\mathbf{M}_n(C)$] and we shall call it the *canonical Riemannian structure*.

The Hodge-star isomorphism can be introduced as usual, i.e.,

$$\star: \quad \Omega_{\mathrm{Der}}{}^{p}(\mathbf{M}_n(C)) \to \Omega_{\mathrm{Der}}{}^{n^2-1-p}(\mathbf{M}_n(C)) \tag{112a}$$

such that, on any p-form α_p given by the product of p basic 1-forms θ^k, it is defined by

$$\star(\theta^{i_1} \wedge \cdots \wedge \theta^{i_p}) = \frac{(|k|)^{1/2}}{(n^2 - 1 - p)!} K^{i_1 j_1} \cdots K^{i_p j_p} \epsilon_{j_1 \cdots j_{n^2-1}} \theta^{j_{p+1}} \wedge \cdots \wedge \theta^{j_{n^2-1}} \tag{112b}$$

and, with respect to the multiplication by *functions* E [elements of $\mathbf{M}_n(C)$], one postulates

$$\star(E\theta^{i_1} \wedge \cdots \wedge \theta^{i_p}) = E[\star(\theta^{i_1} \wedge \cdots \wedge \theta^{i_p})] \tag{112c}$$

where $\epsilon_{j_1 \cdots j_{n^2-1}}$ is the totally antisymmetric Levi-Civita symbol defined as usual, with

$$\epsilon_{123 \cdots (n^2-1)} = 1 \tag{113}$$

Then, one has

$$\star(\Omega_{\mathrm{Der}}{}^{p}(M_n(C))) \subset \Omega_{\mathrm{Der}}{}^{n^2-1-p}(\mathbf{M}_n(C))$$

and

$$\star(\star\alpha_p) = (-1)^{pn^2}\alpha_p \tag{114}$$

In the graded differential algebra $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$, one can also introduce a scalar product between any two *real* exterior differential forms:

$$(\cdot|\cdot): \quad \Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) \times \Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) \to R \tag{115a}$$

defined by

$$(\alpha|\beta) = \begin{cases} \int \alpha \wedge \star(\beta) & \text{if} \quad \alpha, \beta \in \Omega_{\mathrm{Der}}{}^{p}(\mathbf{M}_n(C)) \\ 0 & \text{otherwise (i.e., not of the same degree)} \end{cases} \tag{115b}$$

In view of the graded trace property (see Lemma 1), one has

$$(\alpha|\beta) = (\beta|\alpha) \tag{115c}$$

and this inner product is a *real, positive-definite* bilinear form on the *real* subspace of *real* elements of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$.

This definition can be extended to the *complex* p-forms as

$$\langle\alpha|\beta\rangle = (\alpha^\star|\beta) \tag{115d}$$

where $\langle\cdot|\cdot\rangle$ is a *positive-definite Hermitian* bilinear form on $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$. So, $(\Omega_{\mathrm{Der}}(\mathbf{M}_n(C)); \langle\cdot|\cdot\rangle)$ is a graded finite-dimensional complex Hilbert space. Define now an antidifferentiation

$$\delta\colon \quad \Omega_{\mathrm{Der}}{}^p(\mathbf{M}_n(C)) \to \Omega_{\mathrm{Der}}{}^{p-1}(\mathbf{M}_n(C)) \tag{116a}$$

by

$$\langle d\alpha|\beta\rangle = \langle\alpha|\delta\beta\rangle \tag{116b}$$

$\forall\alpha, \beta \in \Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$.

Integrating by parts and using the fact that $\int$ is *closed* [see equation (109b)], one verifies that

$$\delta(\alpha_p) = (-1)^{(n^2-1)p+n^2}\star \circ d \circ \star(\alpha_p) \in \Omega_{\mathrm{Der}}{}^{p-1}(\mathbf{M}_n(C)) \tag{116c}$$

for any p-form α_p. It follows that, in particular, one has

$$\delta(\alpha_0) = 0 \tag{116d}$$

if α_0 is a 0-form (i.e., a *function*).

As usual, one can now define the Laplace–Beltrami operator (i.e., the Laplacian) Δ on $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ by

$$\Delta = d \circ \delta + \delta \circ d \tag{117a}$$

Using equations (115c) and (116b) and the bilinearity of $\langle\cdot|\cdot\rangle$, one has

$$\langle\alpha|\Delta\alpha\rangle = \langle d\alpha|d\alpha\rangle + \langle\delta\alpha|\delta\alpha\rangle = \|d\alpha\|^2 + \|\delta\alpha\|^2 \tag{117b}$$

It follows that:

(a) Δ is a *definite-positive* operator on the Hilbert space $(\Omega_{\mathrm{Der}}(\mathbf{M}_n(C)$; $\langle\cdot|\cdot\rangle)$:

$$\Delta \geq 0 \tag{117c}$$

(b) We have

$$\Delta\alpha = 0 \Leftrightarrow d\alpha = 0 \text{ and } \delta\alpha = 0 \tag{117d}$$

Any element $\alpha \in \Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ which satisfies (117d) is called *harmonic*. The set of these elements is the kernel of Δ. It is a graded vector space.

All exterior forms satisfying

$$\delta\alpha = 0 \tag{118a}$$

are said to be *orthogonal* in the sense of (115b) to the closed forms β for which

$$d\beta = 0 \tag{118b}$$

This means that, by definition, the *orthogonal complement* of $\delta\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ is the space of forms $\beta \in \Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ satisfying (118b) and the *orthogonal complement* of $d\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ is the space of forms $\alpha \in \Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ satisfying (118a). There follows a *decomposition*

$$\Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) = d\Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) \oplus \delta\Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) \oplus \mathrm{Ker}(\Delta) \tag{118c}$$

of the Grassmannian $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ in three orthogonal subspaces, which is the analog of the Hodge–De Rham decomposition.

Proposition 6. The linear mapping of $\mathrm{Ker}(\Delta)$ in $H_{\mathrm{Der}}(\mathbf{M}_n(C))$ that associates to $\alpha \in \mathrm{Ker}(\Delta)$ its class $[\alpha]$ in $H_{\mathrm{Der}}(\mathbf{M}_n(C))$ is an isomorphism of graded vector spaces. Furthermore, $\alpha \in \mathrm{Ker}(\Delta)$ if and only if α is an invariant element of the subalgebra $\mathbb{1} \otimes \Lambda sl(n; C)^*$ of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ generated by the θ^k, $k \in I = \{1, 2, \ldots, n^2 - 1\}$.

Proof. Let $\alpha \in \mathrm{Ker}(\Delta)$; then if $\alpha + d\beta \in \mathrm{Ker}(\Delta)$, one has [see equation (117d)]

$$\delta d\beta = 0$$

So, from equation (116b) one has

$$\langle \beta | \delta d\beta \rangle = \langle d\beta | d\beta \rangle = \| d\beta \|^2 = 0$$

which implies that

$$d\beta = 0$$

This shows that $\mathrm{Ker}(\Delta) \ni \alpha \to [\alpha] \in H_{\mathrm{Der}}(\mathbf{M}_n(C))$ is injective.

The subalgebra $\mathbb{1} \otimes \Lambda sl(n, C)^* \subset \Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ generated by the θ^k is a differential subalgebra of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$. Let I_0 denote the algebra of the invariant elements of $\mathbb{1} \otimes \Lambda sl(n, C)^*$. By using the Koszul formula (Koszul, 1950) and equations (116), one checks that if $\alpha \in I_0$, one has $d\alpha = 0$ and $\delta\alpha = 0$.

On the other hand, one knows (Koszul, 1950) that $\alpha \to [\alpha]$ is a bijection of I_0 onto $H_{\mathrm{Der}}(\mathbf{M}_n(C)) = H^*(sl(n, C))$. This shows that $\alpha \to [\alpha]$ is *surjective* and therefore *bijective* from $\mathrm{Ker}(\Delta)$ onto $H_{\mathrm{Der}}(\mathbf{M}_n(C))$ and that therefore $\mathrm{Ker}(\Delta)$ coincides with I_0. ■

Remark. The last statements have a classical geometrical interpretation. If one identifies the generators θ^k with the components of the Maurer–Cartan form of $su(n)$, then $\mathbb{1} \otimes \Lambda sl(n, C)^*$ is identified with the differential algebra of left-invariant forms on $su(n)$, and I_0 with the algebra of bi-invariant forms on $su(n)$. Then, the 2-form K given by equation (111) is the metric of $su(n)$ (up to a factor) and the harmonic forms [see equation (117d)] are nothing other than the bi-invariant forms on $su(n)$. These results are also true for any compact semisimple Lie group.

3.2.5. The Example of $\mathbf{M}_2(C)$

The generators of $\mathbf{M}_2(C)$ are the 2×2 unit matrix $\mathbb{1}$ and the Hermitian traceless 2×2 matrices σ_j, $j = 1, 2, 3$ (Pauli matrices) (Dubois-Violette *et al.*, 1990a; Kerner, 1990),

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{119}$$

One has

$$\sigma_k \cdot \sigma_l = \delta_{kl} \cdot 1 + \mathrm{i}\epsilon_{kl}{}^m \sigma_m \tag{120}$$

Thus, comparing to formula (79), one obtains

$$K_{kl} = \delta_{kl} \tag{121a}$$

$$C_{kl}{}^m = C_{klm} = -2\epsilon_{klm} \tag{121b}$$

$$S_{kl}{}^m = 0 \tag{121c}$$

As in Section 3.2.2, one introduces

$$e_k = \mathrm{ad}(i\sigma_k) \tag{122a}$$

and the $\theta^k \in \Omega_{\mathrm{Der}}(\mathbf{M}_2(C))$ such that

$$\theta^k(e_p) = \delta^k_p \mathbb{1} \tag{122b}$$

Then,

$$\sigma_k \theta^l = \theta^l \sigma_k \tag{123}$$

$$\theta^k \wedge \theta^l = -\theta^l \wedge \theta^k \tag{124}$$

$$d\sigma_k = 2\epsilon_{kl}{}^m \sigma_m \theta^p \tag{125}$$

$$d\theta^k = \epsilon^k_{lm} \theta^l \wedge \theta^m \tag{126}$$

The formulas (120) and (123)–(126) give a presentation of $\Omega_{\mathrm{Der}}(\mathbf{M}_2(C))$.

The Hodge-star operator [see equation (112)] acts on the generators of $\Omega_{\mathrm{Der}}(\mathbf{M}_2(C))$ as follows:

$$\star(1) = \frac{1}{6}\epsilon_{klm}\,\theta^k \wedge \theta^l \wedge \theta^m \tag{127a}$$

$$\star(\theta^k) = \frac{1}{2}\epsilon^k_{lm}\,\theta^l \wedge \theta^m \tag{127b}$$

$$\star(\theta^k \wedge \theta^l) = \epsilon^{kl}_m\,\theta^m \tag{127c}$$

$$\star(\theta^k \wedge \theta^l \wedge \theta^m) = \epsilon^{klm}\,\mathbb{1} \tag{127d}$$

Let us now proceed to a *diagonalization* of the Laplacian Δ. Since Δ is *invariant* in the sense that

$$L_\chi \circ \Delta = \Delta \circ L_\chi \tag{128}$$

$\forall\chi \in \mathrm{Der}(\mathbf{M}_2(C))$, it follows that the *irreducible components* of $\Omega_{\mathrm{Der}}(\mathbf{M}_2(C))$ for the representation of $sl(2, C)$ [$\equiv \mathrm{Der}(\mathbf{M}_2(C))$] given by

$$\chi \to L_\chi$$

are the *eigenstates* of Δ. This representation corresponds to the adjoint representation, i.e., of $so(3)$.

Furthermore, one has

$$\star \circ \Delta = \Delta \circ \star \tag{129}$$

so it is sufficient to study only $\Omega_{\mathrm{Der}}{}^0(\mathbf{M}_2(C))$ and $\Omega_{\mathrm{Der}}{}^1(\mathbf{M}_2(C))$.

The first $\Omega_{\mathrm{Der}}{}^0(\mathbf{M}_2(C)) = \mathbf{M}_2(C)$ splits into two irreducible components:

1. The one-dimensional subspace spanned by $\mathbb{1}$.
2. The three-dimensional subspace spanned by the σ_k.

Then, one obtains the following *eigenfunctions* of Δ on $\Omega_{\mathrm{Der}}{}^0(\mathbf{M}_2(C))$:

$$\Delta(\mathbb{1}) = 0 \tag{130a}$$

$$\Delta(\sigma_k) = 8\sigma_k \tag{130b}$$

On the other hand, $\Omega_{\mathrm{Der}}{}^1(\mathbf{M}_2(C))$ splits into four irreducible components:

1. The three-dimensional subspace spanned by the θ^k.
2. The three-dimensional subspace spanned by $\beta^l_k = \sigma_k\theta^l - \sigma_l\theta^k$ or equivalently by the $d\sigma_k$.
3. The five-dimensional subspace spanned by $\rho^l_k = \sigma_k\theta^l + \sigma_l\theta^k - \frac{2}{3}\delta^l_k\sigma_p\theta^p$.
4. The one-dimensional subspace spanned by $\theta = \sigma_k\theta^k$.

In this case, the *eigenfunctions* of Δ are given by

$$\Delta(\theta^k) = 4\theta^k \tag{130c}$$

$$\Delta(d\sigma_k) = 8d\sigma_k \tag{130d}$$

$$\Delta(\rho_k^l) = 16\rho_l^k \tag{130e}$$

$$\Delta(\theta) = 4\theta \tag{130f}$$

For $\Omega_{\text{Der}}{}^2(\mathbf{M}_2(C))$ one takes the star of the decomposition of $\Omega_{\text{Der}}{}^1(\mathbf{M}_2(C))$ and for $\Omega_{\text{Der}}{}^3(\mathbf{M}_2(C))$ one takes the star of the decomposition of $\Omega_{\text{Der}}{}^0(\mathbf{M}_2(C))$.

The obtained *eigenvalues* in equations (130) are 0, 4, 8, and 16. The space of 32 *linearly independent functions* is composed of orthogonal subspaces of the Grassmannian $\Omega_{\text{Der}}(\mathbf{M}_2(C))$ corresponding to different *eigenvalues*.

Likewise, the operator $\delta + d$ can be *diagonalized.* Its *eigenvalues* are 0, ± 2, $\pm(2)^{1/2}$, and ± 4. Of course, the corresponding *eigenfunctions* are no longer homogeneous p-forms.

Finally, similar considerations of invariance and commutation with the star-isomorphism apply for $\mathbf{M}_n(C)$. However, to compute the eigenvalues of Δ, one needs explicitly the coefficients in the formula (79).

3.3. The Case of $A = C^\infty(M) \otimes \mathbf{M}_n(C)$

Let us investigate now the possibility of constructing a larger algebraic structure which would contain a *commutative* part and a *noncommutative* part. The most natural choice is to consider the algebra $A = A_0 \otimes \mathbf{M}_n(C)$ of smooth $M_n(\mathbf{C})$-valued functions on a connected and simply connected manifold M, where $A_0 = C^\infty(M)$ represents the *commutative* part (see Section 3.1) and $\mathbf{M}_n(C)$ the *noncommutative* part (see Section 3.2) (Dubois-Violette *et al.*, 1990b). In this section, we will study the noncommutative differential geometry of A. Some features of the noncommutative geometry of algebras of this type were investigated in Madore (1988; see also Madore 1993a,b) in a different context.

3.3.1. Differential Calculus and Presentation of $\Omega_{Der}(A)$

At any point $x \in M$, one may define a homomorphism of $C^\star$-algebras with units

$$h_x\colon \quad A = A_0 \otimes \mathbf{M}_n(C) \to \mathbf{M}_n(C)$$

by

$$h_x(f \otimes E) = f(x)E \tag{131}$$

$\forall f \in A_0$ and $\forall E \in \mathbf{M}_n(C)$.

The center of A is its subalgebra $A_0 \otimes \mathbb{1}$ and the Lie algebra Der(A) of all derivations of A is a module over its center. So, Der(A) is an A_0-module.

In general, for two arbitrary associative algebras A_1 and A_2, the derivation of their tensor product is not the simple sum of derivations of each of them:

$$\mathrm{Der}(A_1 \otimes A_2) \neq \mathrm{Der}(A_1) \oplus \mathrm{Der}(A_2)$$

In our case, where Der(A_0) is the Lie algebra of smooth vector fields ∂_μ on M and Der($\mathbf{M}_n(C)$) is the Lie algebra $sl(n;\, C)$, it is clear that

$$(\mathrm{Der}(A_0) \otimes \mathbb{1}) \oplus (A_0 \otimes \mathrm{Der}(\mathbf{M}_n(C)))$$

is a Lie subalgebra and an A_0-module of Der(A). In fact, one has the following result.

Lemma 2.

$$\mathrm{Der}(A) = (\mathrm{Der}(A_0) \otimes \mathbb{1}) \oplus (A_0 \otimes \mathrm{Der}(\mathbf{M}_n(C))) \tag{132}$$

Proof. Let χ be a derivation of A. Then,

$$A_0 \ni f \to \chi(f \otimes \mathbb{1})$$

is a $\mathbf{M}_n(C)$-valued vector field on M. Therefore, one has

$$\chi(f \otimes E) = \chi((f \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes E)) = \chi((\mathbb{1} \otimes E) \cdot (f \otimes \mathbb{1})) \tag{133a}$$

i.e.,

$$\chi(f \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes E) + (f \otimes \mathbb{1}) \cdot \chi(\mathbb{1} \otimes E) = (\mathbb{1} \otimes E) \cdot \chi(f \otimes \mathbb{1}) + \chi(\mathbb{1} \otimes E) \cdot (f \otimes \mathbb{1}) \tag{133b}$$

and therefore

$$\chi(f \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes E) = (\mathbb{1} \otimes E) \cdot \chi(f \otimes \mathbb{1}) \tag{133c}$$

$\forall f \in A_0$ and $\forall E \in \mathbf{M}_n(C)$. It follows that $\chi(f \otimes \mathbb{1})$ is in $A_0 \otimes \mathbb{1}$, $\forall f \in A_0$. This shows that the restriction $\chi_{|(A_0 \otimes \mathbb{1})}$ is in Der(A_0) $\otimes\ \mathbb{1}$.

Now, at any point $x \in M$, the mapping

$$E \to h_x(\chi(\mathbb{1} \otimes E))$$

defines a derivation of $\mathbf{M}_n(C)$, where h_x is given by equation (131).

This implies that the restriction $\chi_{|(\mathbb{1} \otimes \mathbf{M}_n(C))}$ is in $A_0 \otimes$ Der($M_n(C)$). ∎

An element a of A is a *generalized function* in the sense that it generalizes the notion of a smooth function f on M into a notion of a smooth $\mathbf{M}_n(C)$-valued function on M. A *presentation* of $\Omega_{\mathrm{Der}}(A)$ is essentially based on the *presentation* of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ given by the relations (79), (94), (95), (96a), and (101) in Section 3.2.2. Hence, in a given basis $\{\mathbb{1}, E_k\}$, $k \in I = \{1, 2, \ldots, n^2 - 1\}$, one has

$$a = f^0(x) \otimes \mathbb{1} + f^k(x) \otimes E_k = f^0(x)\mathbb{1} + f^k(x)E_k \tag{134}$$

Then, any derivation χ of A can be represented as

$$\chi = \chi^{\mu}(x)\, \partial_{\mu} \otimes \mathbb{1} + \chi^k(x) \otimes \mathrm{ad}(\mathrm{i}E_k) \tag{135}$$

where $\partial_{\mu} = \partial/\partial x^{\mu} \in \mathrm{Der}(A_0)$, $\mu = 1, \ldots, \dim(M) = m$, $\mathrm{ad}(\mathrm{i}E_k) = e_k \in \mathrm{Der}(\mathbf{M}_n(C))$, and $\chi^{\mu}(x)$ and $\chi^k(x)$ are functions of $x \in M$.

Now, define the graded differential algebra $\Omega_{\mathrm{Der}}(\mathrm{A})$. First, recall that for any two arbitrary graded differential algebras Ω_1 and Ω_2 with differentials d_1 and d_2, respectively, the tensor product $\Omega_1 \otimes \Omega_2$ is naturally a graded differential algebra with differential d if the tensor product is defined by

$$(\alpha \otimes \beta) \wedge (\eta \otimes \rho) = (-1)^{rs}(\alpha \wedge \eta) \otimes (\beta \wedge \rho) \tag{136a}$$

for $\alpha \in \Omega_1$, $\beta \in \Omega_2^r$, $\eta \in \Omega_1^s$, and $\rho \in \Omega_2$, and the differential d by

$$d(\alpha \otimes \beta) = (d_1\alpha) \otimes \beta + (-1)^P \alpha \otimes (d_2\beta) \tag{136b}$$

$\forall \alpha \in \Omega_1^p$ and $\forall \beta \in \Omega_2$.

It follows from Lemma 2 that $C(\mathrm{Der}(A_0); A_0) \otimes C(\mathrm{Der}(\mathbf{M}_n(C)); \mathbf{M}_n(C))$ is a graded differential subalgebra of $C(\mathrm{Der}(A); A)$. Then, the smallest differential subalgebra $\Omega_{\mathrm{Der}}(A)$ of the complex $C(\mathrm{Der}(A); A)$ which contains A is

$$\Omega_{\mathrm{Der}}(A) = \Omega_{\mathrm{Der}}(A_0) \otimes \Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) = \Omega(M) \otimes \mathbf{M}_n(C) \otimes \Lambda sl(n; C)^* \tag{137}$$

where $\Omega_{\mathrm{Der}}(A_0) = \Omega(M)$ is the graded differential algebra of exterior differential forms on M identified with the differential subalgebra $\Omega(M) \otimes \mathbb{1}$ of $\Omega_{\mathrm{Der}}(A)$, and $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$, given by equation (74), is identified with the differential subalgebra $\mathbb{1} \otimes \Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ of $\Omega_{\mathrm{Der}}(A)$. In the second (or the third) member of equation (137), the tensor product is the usual (twisted) tensor product of graded differential algebras.

Let us remark that, generally, for two arbitrary associative algebras A_1 and A_2 one has

$$\Omega_{\mathrm{Der}}(A_1 \otimes A_2) \neq \Omega_{\mathrm{Der}}(A_1) \otimes \Omega_{\mathrm{Der}}(A_2)$$

For instance, for $A_1 = \mathbf{M}_n(C)$ and $A_2 = \mathbf{M}_m(C)$ one has

$$\Omega_{\mathrm{Der}}(\mathbf{M}_n(C) \otimes \mathbf{M}_m(C)) = \mathbf{M}_n(C) \otimes \mathbf{M}_m(C) \otimes \Lambda sl(nm; C)^*$$

when

$$\begin{aligned} &\Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) \otimes \Omega_{\mathrm{Der}}(\mathbf{M}_m(C)) \\ &= \mathbf{M}_n(C) \otimes \Lambda sl(n; C)^* \otimes M_m(C) \otimes \Lambda sl(m; C)^* \end{aligned}$$

and

$$sl(nm; C) = (sl(n; C) \otimes sl(m; C)) \oplus (sl(n; C) \otimes \mathbb{1}) + (\mathbb{1} \otimes sl(m; C))$$

Then, the relation (137) is particular enough in the sense that it comes from the commutativity of $C^\infty(M)$. Now, from equation (137) one deduces that $\Omega_{\text{Der}}(A)$ is naturally a *bigraded differential algebra*:

$$\Omega^{r,s}_{\text{Der}}(A) = \Omega^r(M) \otimes \Omega^s_{\text{Der}}(\mathbf{M}_n(C)) \tag{138}$$

If d denotes the differential of $\Omega_{\text{Der}}(A)$, then

$$d = d_1 + d_2 \tag{139a}$$

where d_1 is the unique antiderivation of $\Omega_{\text{Der}}(A)$ extending the exterior differential of $\Omega(M)$ such that

$$d_1(\Omega_{\text{Der}}(\mathbf{M}_n(C))) = 0 \tag{139b}$$

and d_2 is the unique antiderivation of $\Omega_{\text{Der}}(A)$ extending the differential of $\Omega_{\text{Der}}(\mathbf{M}_n(C))$ such that

$$d_2(\Omega(M)) = 0 \tag{139c}$$

The bidegrees of d_1, d_2, and d are (1, 0), (0, 1), and (1, 1), respectively, and one has

$$d^2 = d_1^2 = d_2^2 = d_1 d_2 + d_2 d_1 = 0 \tag{139d}$$

Then, $\Omega_{\text{Der}}(A)$ is effectively a bigraded differential algebra and the action of the differential d on a *generalized function* $a \in A$ is given by

$$\begin{aligned} da &= d_1 f^o(x) \otimes \mathbb{1} + d_1 f^k(x) \otimes E_k + f^k(x) \otimes d_2 E_k \\ &= [\partial_\mu f^o(x) \otimes \mathbb{1} + \partial_\mu f^k(x) \otimes E_k] d_1 x^\mu - [f^k(x) C^m_{kl} \otimes E_m]\, \theta^l \end{aligned} \tag{140}$$

Finally, we close this subsection with a discussion of the notion of reality in the case of the $\star$-algebra $A = A_0 \otimes \mathbf{M}_n(C)$. Let $\text{Der}_{\mathbf{R}}(A)$ denote the real Lie subalgebra of $\text{Der}(A)$ of derivations χ such that [equations (44) and (48)]

$$\chi(a^\star) = (\chi(a))^\star \tag{141}$$

In our case, $\text{Der}_{\mathbf{R}}(A)$ is defined by

$$\text{Der}_{\mathbf{R}}(A) = (\text{Der}_{\mathbf{R}}(A_0) \otimes \mathbb{1}) \oplus (C^\infty_{\mathbf{R}}(M) \otimes \text{Der}_{\mathbf{R}}(\mathbf{M}_n(C)) \tag{142}$$

where $\text{Der}_{\mathbf{R}}(A_0)$ is the real Lie algebra of real vector fields on M, $C^\infty_{\mathbf{R}}(M)$ is the real algebra of real functions on M, and $\text{Der}_{\mathbf{R}}(\mathbf{M}_n(C))$ is the Lie algebra $su(n)$ for its adjoint action on $\mathbf{M}_n(C)$.

The extension of this antilinear involution on $\Omega_{\mathrm{Der}}(A)$ is defined by

$$(\alpha \otimes \beta)^{\star} = \alpha^{\star} \otimes \beta^{\star} \tag{143a}$$

for $\alpha \in \Omega(M)$ and $\beta \in \Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ with

$$\alpha \to \alpha^{\star}$$

being the usual *complex conjugation* of differential forms on M and

$$\beta \to \beta^{\star}$$

being the *involution* of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ [see equation (102)].

An element ω of $\Omega_{\mathrm{Der}}(A)$ is then said to be *real* if

$$\omega^{\star} = \omega \tag{143b}$$

and *purely imaginary* if

$$\omega^{\star} = -\omega \tag{143c}$$

3.3.2. Metric for A and scalar product for $\Omega_{Der}(A)$

Let M be an oriented Riemannian manifold with a metric ds^2 defined in a system of local coordinates $\{x^{\mu}\}$ by

$$ds^2 = g_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \tag{144}$$

and let $\{g^{\mu\nu}\}$ be the inverse matrix of $g_{\mu\nu}$:

$$g_{\mu\nu} g^{\nu\sigma} = \delta^{\sigma}_{\mu} \tag{145}$$

On the other hand, we introduced in Section 3.2.4 the so-called *canonical Riemannian structure* for $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ [see equation (111)]:

$$K = K_{kl} \theta^k \wedge \theta^l$$

where K_{kl} is the Cartan–Killing metric for $su(n)$.

The *metric G* for *A* is naturally obtained by a combination of these two structures, i.e.,

$$G = g_{\mu\nu} dx^{\mu} \wedge dx^{\nu} + \rho^2 K_{mn} \theta^m \wedge \theta^n \tag{146}$$

where ρ is some constant that may be interpreted as a length [or $(\text{mass})^{-1}$, denoted by $\rho = m^{-1}$, in the case where $\hbar = c = 1$].

Now, define a *scalar product* for $\Omega_{\mathrm{Der}}(A) = \Omega(M) \otimes \Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$. First, the scalar product for $\Omega(M)$ is defined by means of the Hodge-star isomorphism:

$$\star\colon \quad \Omega^p(M) \to \Omega^{m-p} \tag{147}$$

associated with the metric (144) and the orientation of M^m, such that

$$\langle\alpha|\alpha'\rangle = \begin{cases} \int_M \alpha^\star \wedge \star(\alpha') & \text{if} \quad \alpha \text{ and } \alpha' \in \Omega^p(M) \\ 0 & \text{otherwise} \quad \text{(i.e., if not of the same degree)} \end{cases} \tag{148}$$

At this level, let us remark that this *positive Hermitian scalar product* is defined on $\Omega(M)$ only if M is *compact*. Otherwise, one has to restrict attention to differential forms for which $\langle\alpha|\alpha\rangle < \infty$, for example, the forms with compact support.

In Section 3.2.4, we introduced a Hodge-star isomorphism of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ [see equations (112a)–(112c)] that is associated with the canonical Riemannian structure K. Using this Hodge-star isomorphism and the graded trace property (see Lemma 1), we defined a *scalar product* on $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ [see equations (115a)–(115d)].

In the case of $\Omega_{\mathrm{Der}}(A)$, the *metric* is now defined by (146) and it follows from the presence of the constant ρ^2 that the *scalar product* on $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ given by equation (115b) is rewritten as

$$\langle\beta|\beta'\rangle = \begin{cases} \rho^{n^2-1-2p} \int \beta^\star \wedge \star(\beta') & \text{if} \quad \beta, \beta' \in \Omega^p_{\mathrm{Der}}(\mathbf{M}_n(C)) \\ 0 & \text{otherwise} \end{cases} \tag{149}$$

Now, by means of equations (148) and (149) one may define a *scalar product* on $\Omega_{\mathrm{Der}}(A)$ by

$$\langle\alpha \otimes \beta|\alpha' \otimes \beta'\rangle = \langle\alpha|\alpha'\rangle \cdot \langle\beta|\beta'\rangle \tag{150a}$$

$\forall\alpha, \alpha' \in \Omega(M)$ and $\beta, \beta' \in \Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$, i.e.,

$$\langle\alpha \otimes \beta|\alpha' \otimes \beta'\rangle \tag{150b}$$

$$= \begin{cases} \rho^{n^2-1-2q} \int_M \alpha^\star \wedge \star(\alpha') \cdot \int \beta^\star \wedge \star(\beta') \\ \quad \text{if} \quad \alpha, \alpha' \in \Omega^p(M) \quad \text{and} \quad \beta, \beta' \in \Omega^q_{\mathrm{Der}}(\mathbf{M}_n(C)) \\ 0 \quad \text{otherwise} \end{cases}$$

4. NONCOMMUTATIVE SYMPLECTIC STRUCTURES

4.1. Introduction

Let M be a smooth manifold. Recall that a *symplectic form* on M is a real, closed, nondegenerate differential 2-form ω on M. Therefore, ω is a *symplectic structure for the commutative algebra* $A_0 = C^\infty(M)$ of smooth complex functions on M if and only if (M, ω) is a *symplectic manifold* (Guillemin and Sternberg, 1984; Arnold, 1989).

Given such a 2-form, one defines the *Hamiltonian* vector field $\mathrm{Ham}(f)$ associated with $f \in A_0$ by

$$\omega(\chi, \mathrm{Ham}(f)) = \chi(f) \tag{151}$$

for any vector field $\chi \in V(M) \equiv \mathrm{Der}(A_0)$, and one defines the *Poisson bracket* $\{f, g\}_P$ of two functions $f, g \in A_0$ by

$$\{f, g\}_P = \omega(\mathrm{Ham}(f), \mathrm{Ham}(g)) = \mathrm{Ham}(f)(g) = -\mathrm{Ham}(g)(f) = -\{g, f\}_P \tag{152}$$

The closure property

$$d\omega(\mathrm{Ham}(f), \mathrm{Ham}(g), \mathrm{Ham}(h)) = 0 \tag{153a}$$

induces the Jacobi identity

$$\{f, \{g, h\}_P\}_P + \{h, \{f, g\}_P\}_P + \{g, \{h, f\}_P\}_P = 0 \tag{153b}$$

and the property

$$d\omega(\chi, \mathrm{Ham}(f), \mathrm{Ham}(g)) = 0 \tag{154a}$$

$\forall \chi \in \mathrm{Der}(A_0)$ is equivalent to

$$[\mathrm{Ham}(f), \mathrm{Ham}(g)] = \mathrm{Ham}(\{f, g\}_P) \tag{154b}$$

where [,] is the Lie bracket. Furthermore, it follows from

$$\{f, g\}_P = \mathrm{Ham}(f)(g) \tag{155}$$

that one has

$$\{f, g \cdot h\}_P = \{f, g\}_P \cdot h + g \cdot \{f, h\}_P \tag{156}$$

4.2. Definitions

Let us now define the analog of the *classical* symplectic structure in the case of a noncommutative associative algebra A.

Definition 1. An element ω of the graded differential algebra $\Omega^2_{\mathrm{Der}}(A)$ will be called a *symplectic structure* for A if and only if it satisfies the following conditions:

(a) *Nondegeneration:* For any $a \in A$, there is a derivation $\mathrm{Ham}(a) \in \mathrm{Der}(A)$ such that

$$\omega(\chi, \mathrm{Ham}(a)) = \chi(a) \tag{157}$$

for any $\chi \in \mathrm{Der}(A)$.

(b) *Closure:* ω is closed, i.e.,

$$d\omega = 0 \tag{158}$$

Notice that condition (a) implies that $\text{Ham}(a) \in \text{Der}(A)$ is unique for a given $a \in A$, i.e., one has a linear mapping

$$A \to \text{Der}(A) \tag{159}$$

Remark also that in the commutative case $A_0 = C^\infty(M)$, condition (a) means that ω is a *nondegenerate* 2-form on M.

Definition 2. Let ω be a symplectic structure for A. One defines the corresponding *generalized Poisson bracket* $\{a, b\}$ of $a, b \in A$ by

$$\{a, b\} = \omega(\text{Ham}(a), \text{Ham}(b)) \tag{160}$$

It follows that

$$\{a, b\} = -\{b, a\} \tag{161}$$

and that the mapping

$$b \to \{a, b\} \tag{162}$$

is a derivation of A which is precisely $\text{Ham}(a)$.

Furthermore, condition (b) in Definition 1 implies that this Poisson bracket satisfies the Jacobi identity

$$\{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0 \tag{163}$$

and

$$[\text{Ham}(a), \text{Ham}(b)] = \text{Ham}(\{a, b\}) \tag{164}$$

Thus, everything works as in the commutative case (see Section 4.1).

5. EXAMPLES

5.1. The Case of $A = \mathbf{M}_n(C)$ (The Canonical Symplectic Structure)

Here, the procedure is based on the fact that $A = \mathbf{M}_n(C)$ is the analog of $A_0 = C^\infty(M)$, $\text{Der}(A)$ is the analog of the Lie algebra of vector fields $V(M) = \text{Der}(A_0)$, and $\Omega_{\text{Der}}(A)$ is the analog of the algebra of differential forms $\Omega(M)$.

In this case, it is obvious to call a *symplectic structure* a real closed element ω of $\Omega^2_{\text{Der}}(\mathbf{M}_n(C))$ such that, for each $E \in \mathbf{M}_n(C)$,

$$\omega(\chi, \text{Ham}(E)) = \chi(E) \tag{165}$$

$\forall \chi \in \text{Der}(\mathbf{M}_n(C))$, possesses a unique solution $\text{Ham}(E) \in \text{Der}(\mathbf{M}_n(C))$.

Then, the *Poisson bracket* $\{E, F\}$ of $E, F \in \mathbf{M}_n(C)$ is defined by

$$\{E, F\} = \omega(\mathrm{Ham}(E), \mathrm{Ham}(F)) = -\{F, E\} \tag{166}$$

Furthermore, the property

$$d\omega\,(\mathrm{Ham}(E), \mathrm{Ham}(F), \mathrm{Ham}(G)) = 0 \tag{167a}$$

is equivalent to the Jacobi identity

$$\{E, \{F, G\}\} + \{G, \{E, F\}\} + \{F, \{G, E\}\} = 0 \tag{167b}$$

and

$$[\mathrm{Ham}(E), \mathrm{Ham}(F)] = \mathrm{Ham}\{\{E, F\}\} \tag{167c}$$

Thus, all works as in the classical case (see Section 4.1).

Suppose now that there is a symplectic structure for $A = \mathbf{M}_n(C)$. Then, its Poisson bracket $\{\,,\}$ must be proportional to the *commutator* $[\,,]_-$ since it is a derivation in each variable which is antisymmetric and since all derivations are inner. Thus, one must have

$$\{E, F\}_\hbar = \frac{\mathrm{i}}{\hbar}\,[E, F]_-$$

$\forall E, F \in \mathbf{M}_n(C)$ and $\hbar$ is some number.

On the other hand, since $\Omega_{\mathrm{Der}}{}^2(\mathbf{M}_n(C)) = C^2[\mathrm{Der}(\mathbf{M}_n(C); \mathbf{M}_n(C))$, one defines an element ω of $\Omega_{\mathrm{Der}}{}^2(\mathbf{M}_n(C))$ by setting

$$\omega\left(\mathrm{ad}\left(\frac{i}{\hbar}\,E\right), \mathrm{ad}\left(\frac{i}{\hbar}\,F\right)\right) = \mathrm{ad}\left(\frac{i}{\hbar}\,E\right)(F) = -\mathrm{ad}\left(\frac{i}{\hbar}\,F\right)(E) = \frac{i}{\hbar}\,[E, F]_- \tag{168}$$

$\forall E, F \in \mathbf{M}_n(C)$. This implies that:

(i) ω is a symplectic structure for $\mathbf{M}_n(C)$.

(ii) We have

$$\mathrm{Ham}(E) = \mathrm{ad}\left(\frac{i}{\hbar}\,E\right) \tag{169}$$

(iii) The following holds:

$$\{E, F\}_\hbar = \frac{i}{\hbar}\,[E, F]_- \tag{170}$$

The symplectic structure ω must be *exact* since it is *invariant* and *closed*, i.e., since $H_{\mathrm{Der}}{}^2(\mathbf{M}_n(C)) = 0$ [see equation (76a)]. For this reason and based upon the results of Section 2.5.2, we will define what we will call the *canonical symplectic structure*.

Let

$$\theta = E_k\theta^k = \theta^k E_k \tag{171}$$

be the *canonical invariant element* of $\Omega_{\text{Der}}{}^1(\mathbf{M}_n(C))$ introduced below [see equations (104), (105), (106a), and (106b)]. In fact, θ is independent of the choice of the basis $\{E_k\}$, $k \in I = \{1, 2, \ldots, n^2 - 1\}$, and any scalar multiple of θ is also an *invariant element* of $\Omega_{\text{Der}}{}^1(\mathbf{M}_n(C))$. Let us take the element $\hbar\theta$ and show that it is *invariant* in the sense that

$$L_{e_k}(\hbar\theta) = 0 \tag{172}$$

To show this, it is easy to see that

$$i_{e_k}(\hbar\theta) = \hbar E_k \tag{173}$$

and that the differential of this element does not vanish [see equation (106b)]:

$$d(\hbar\theta) = d(\hbar E_k\theta^k) = \hbar dE_k \wedge \theta^k + \hbar E_k d\theta^k = \frac{\hbar}{2} C_{lm}{}^k E_k\theta^l \wedge \theta^m = i\hbar\theta \wedge \theta \tag{174}$$

where we have used equations (82), (96b), and (101).

So, one has

$$d \circ i_{e_k}(\hbar\theta) = -\hbar C_{kl}{}^m E_m\theta^l \tag{175}$$

Thus,

$$i_{e_k} \circ d(\hbar\theta) = \hbar C_{kl}{}^m E_m\theta^l \tag{176}$$

and finally, one verifies equation (172):

$$L_{e_k}(\hbar\theta) = (d \circ i_{e_k} + i_{e_k} \circ d)(\hbar\theta) = 0$$

Setting by definition

$$\omega = d(\hbar\theta) = \frac{\hbar}{2} C_{kl}{}^m E_m\theta^k \wedge \theta^l \tag{177}$$

it is obvious to see that ω is closed:

$$d\omega = d^2(\hbar\theta) = 0 \tag{178}$$

and *nondegenerate* in the usual sense, i.e., if, for some $e_k = \text{ad}((i/\hbar)E_k) \in \text{Der}(\mathbf{M}_n(C))$, one has

$$\omega(e_k, e_l) = e_k(E_l) = \frac{i}{\hbar}[E_k, E_l]_- = \frac{i}{\hbar} C_{kl}{}^m E_m = 0 \tag{179}$$

for all $e_l = \text{ad}((i/\hbar)E_l) \in \text{Der}(\mathbf{M}_n(C))$, then e_k must be identically zero.

Then, the element $\omega \in \Omega_{\mathrm{Der}}{}^2(\mathbf{M}_n(C))$ given by equation (177) satisfies the conditions of the Definition 1 in Section 4.2 with a unique solution:

$$\mathrm{Ham}(E) = \mathrm{ad}\left(\frac{i}{\hbar}\, E\right) \tag{180}$$

for $E \in \mathbf{M}_n(C)$. This is due precisely to the fact that ω is *nondegenerate*.

Thus, the corresponding *Poisson bracket* is given by

$$\{E, F\}_\hbar = \omega(\mathrm{Ham}(E), \mathrm{Ham}(F)) = \frac{i}{\hbar}\, [E, F]_- \tag{181}$$

for $E, F \in \mathbf{M}_n(C)$. We call ω the *canonical symplectic structure* for $\mathbf{M}_n(C)$.

5.2. The Case of $A = A_\hbar$ (The Heisenberg Algebra)

Define the Heisenberg algebra $A_\hbar$ as the *-algebra with unit generated by two Hermitian elements $\mathbf{p}$ and $\mathbf{q}$ satisfying

$$[\mathbf{p}, \mathbf{q}]_- = i\hbar \mathbb{1} \tag{182}$$

Here, we consider only one degree of freedom for notational convenience, but it is straightforward to take the extension to a finite number of degrees of freedom.

It is easy to show that all elements of $\mathrm{Der}(A_\hbar)$ are inner derivations, so again, as in the above example, if there is a *symplectic structure* for $A_\hbar$ then the corresponding *Poisson bracket* must also be proportional to the commutator (Dirac, 1926)

$$\{A, B\}_\hbar = \frac{i}{\hbar}\, [A, B]_- \tag{183}$$

for A and $B \in A_\hbar$.

On the other hand, we define an element $\omega \in C^2(\mathrm{Der}(A_\hbar); A_\hbar)$ by setting

$$\omega\left(\mathrm{ad}\left(\frac{i}{\hbar}\, A\right), \mathrm{ad}\left(\frac{i}{\hbar}\, B\right)\right) = \{A, B\}_\hbar = \frac{i}{\hbar}\, [A, B]_- \tag{184}$$

Now, to obtain the expression for ω one needs to use the notion of *star-product* in the context of the deformation of algebraic structures (see, for instance, Vey, 1975; Flato *et al.*, 1975, 1976; Bayen *et al.*, 1978). The trick is to remark that (see Proposition 5 in the appendix)

$$\star_0 = \star_{(\star(\cdot;\nu;\{D_1^i\},\{D_2^j\});-\nu;\{D_1^i\},\}D_2^j\})} = \text{``}\cdot\text{''} \tag{185}$$

where

$$\nu = \frac{\hbar}{2i} \tag{186}$$

Then, one directly verifies that ω is given by an infinite sum:

$$\omega = \sum_{n\geq 0} \frac{1}{(2\hbar)^n(n+1)!} \underbrace{[\cdots[d\mathbf{p}, \mathbf{p}], \mathbf{p}, \ldots, \mathbf{p}]}_{n \text{ times}} \wedge \underbrace{[\cdots[d\mathbf{q}, \mathbf{q}], \mathbf{q}, \ldots, \mathbf{q}]}_{n \text{ times}} \tag{187}$$

where only a finite number of terms are different from zero, because we consider only the inferior degree of polynomials whose derivations are ad-type.

This symplectic structure will represents the *quantum analog* of the classical one:

$$\omega_0 = d\mathbf{p} \wedge d\mathbf{q} \tag{188}$$

since

$$[\cdots[d\mathbf{p}, \mathbf{p}], \mathbf{p}, \ldots, \mathbf{p}] \wedge [\cdots[d\mathbf{q}, \mathbf{q}], \mathbf{q}, \ldots, \mathbf{q}] \sim \hbar^{2n} \tag{189}$$

produces the expected *formal classical limit*:

$$\lim_{\hbar\to 0} \omega = \omega_0 \tag{190}$$

Thus, ω is in $\Omega_{\mathrm{Der}}{}^2(A_\hbar)$ and more precisely in its *completion* $\hat{\Omega}_{\mathrm{Der}}{}^2(A_\hbar)$ (see Section 2.5.2) and the properties of the commutator $[,]_-$ induce that ω is indeed a symplectic structure, i.e., satisfies the conditions of Definition 1 in Section 4.2. Here, one has

$$\mathrm{Ham}(A) = \mathrm{ad}\left(\frac{i}{\hbar} A\right) \tag{191}$$

and the corresponding *Poisson bracket* is given by equation (183).

Finally, notice that we do not know if the symplectic structure ω is *exact*, but we argue that it is so because there is no trace on $A_\hbar$.

5.3. The Case of $A = A_0 \otimes \mathbf{M}_n(C)$

Let us consider here the case of an algebra $A_0 = C^\infty(M)$ with (M^{2m}, ω) being a symplectic manifold with local coordinates (q^i, p_j), $i, j = 1, \ldots, m$, and with a symplectic 2-form given by

$$\omega_0 = dp_i \wedge dq^i \tag{192}$$

The algebra A_0 which is used to describe the Hamiltonian classical mechanics (Guillemin and Sternberg, 1984; Arnold, 1989) is a $C^\star$-algebra of smooth functions on M equipped with a pointwise product "·" and a Poisson bracket $\{ , \}_P$ defined by

$$\{f, g\}_P = \frac{\partial f}{\partial p_i} \cdot \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \cdot \frac{\partial g}{\partial p_i} \tag{193}$$

for f, $g \in A_0$, and which satisfies the Jacobi identity and the Leibnitz rule with respect to the pointwise product "·" of functions.

On the other hand, we have already introduced a symplectic structure ω on $\mathbf{M}_n(C)$ defined by [see equations (169) and (170)]

$$\{E, F\}_\hbar = \omega(\mathrm{Ham}(E), \mathrm{Ham}(F)) = \frac{i}{\hbar} [E, F]_- \tag{194}$$

with

$$\mathrm{Ham}(E) = \mathrm{ad}\left(\frac{i}{\hbar} E\right) \tag{195}$$

for E, $F \in \mathbf{M}_n(C)$.

Now, the purpose of this subsection is to define an *extension* of the bracket (193) which would be appropriate also for the *generalized functions* which are in our case just $\mathbf{M}_n(C)$-valued functions on the symplectic manifold M. This *generalized bracket* will be denoted by $\{ , \}$.

Let a, b be two such *generalized functions* ($\in A$) [see equation (134)]:

$$\begin{aligned} a(q, p) &= f^0(q, p) \otimes 1 + f^k(q, p) \otimes E_k \\ b(q, p) &= g^0(q, p) \otimes 1 + g^k(q, p) \otimes E_k \end{aligned} \tag{196}$$

where $f^0, f^k, g^0, g^k \in A_0$ and $\{1, E_k\}$, $k \in I = \{1, \ldots, n^2 - 1\}$, is a basis of $\mathbf{M}_n(C)$.

The most general antisymmetric ad-invariant combination of the two brackets (193) and (194) is the following *generalized bracket*:

$$\begin{aligned} \{a, b\} = {} & (\alpha_1\{f^0, g^0\}_P + \alpha_2\{f^i, g^j\}_P \cdot K_{ij}) \otimes 1 \\ & + (\alpha_3\{f^0, g^k\}_P + \alpha_4\{f^k, g^0\}_P + \alpha_5\{f^i, g^j\}_P \cdot S_j^{ik} \\ & + \alpha_6 f^i \cdot g^j \cdot C_{ij}{}^k) \otimes E_k \end{aligned} \tag{197}$$

where $\alpha_1, \ldots, \alpha_6$ are some constant coefficients to be determined.

Assuming that this generalized bracket obeys a Jacobi identity and using the fact that A_0 is commutative and associative, and together with equations (79)–(83b), (86a), (86b), (153b), (156), (167b), and (170), we find that

$$\alpha_2 = \alpha_5 = 0, \qquad \alpha_1 = \alpha_3 = \alpha_4 = \text{const} \neq 0, \qquad \alpha_6 = \text{const} \neq 0 \tag{198a}$$

Let us choose, for instance,

$$\alpha_1 = \alpha_3 = \alpha_4 = 1 \qquad \text{and} \qquad \alpha_6 = \mathrm{i} \tag{198b}$$

then, the final and unique (up to a constant) expression of this bracket is

$$\{a, b\} = \{f^0, g^0\}_P \otimes \mathbb{1} + (\{f^0, g^k\}_P + \{f^k, g^0\}_P + \mathrm{i} f^{\mathrm{i}} \cdot g^j \cdot C_{\mathrm{i}j}{}^k) \otimes E_k \tag{199}$$

Because the elements of $A = A_0 \otimes \mathbf{M}_n(C)$ are just $\mathbf{M}_n(C)$-valued functions, we expect that the only bracket that can be defined on A is the commutator $[\,,]_-$, exactly as in the case $A = \mathbf{M}_n(C)$ (see Section 3.2).

Effectively, we can easily verify that the Leibnitz rule with respect to the associative multiplication is not satisfied. On the other hand, we have

$$\{a, [b, c]_-\} = [\{a, b\}, c]_- + [b, \{a, c\}]_- \tag{200a}$$

and

$$[a, \{b, c\}]_- = \{[a, b]_-, c\} + \{b, [a, c]_-\} \tag{200b}$$

which means that the generalized bracket $\{\,,\}$ is a derivation of the Lie algebra A equipped with the commutator $[\,,]_-$ and vice versa.

The Leibnitz rule with respect to multiplication can be recovered if we assume that the generalized functions take their values in the *enveloping algebra* $\mathcal{M}_n(C)$ of $\mathbf{M}_n(C)$, i.e., if we consider all the possible products of generators of the type $E_i \cdot E_j$, $E_i \cdot E_j \cdot E_k$, etc., as new elements of the algebra (which now becomes infinite-dimensional), instead of being reduced to expressions linear in $\mathbb{1}$ and E_i. Then, the new bracket with functions whose expressions are linear in $\mathbb{1}$, E_i, $E_i \cdot E_j$, $E_i \cdot E_j \cdot E_k$, etc., will be defined precisely in the way that ensures that the Leibnitz rule holds, but the price to pay is the infinite extension of our noncommutative algebra.

6. MODULES, CONNECTIONS, CURVATURES, AND GAUGE THEORIES

In a rigorous mathematical formulation, gauge theories are nothing else than descriptions of geometrical and dynamical properties of connections and curvatures in the appropriate fiber bundles over the space-time. The fibers F are often interpreted as *internal spaces* on which a given symmetry group acts *transitively* and *effectively.*

In the same spirit, we should replace the internal space by the noncommutative geometry based on some associative algebra A. The advantage here is the absence of an infinite tower of excitations of the internal space, which now become discrete.

The central object of a gauge theory is the connection, or the covariant derivative induced by it. This latter is defined on the space of sections of the corresponding fiber bundle. The set of these sections forms a module over the algebra of C^∞-functions on the base space M with values in some representation of the structural group G. The property of the covariant derivative that is most important is the Leibniz rule with respect to the (right-) multiplication in this module:

Let ϕ be a section, U a group-representation-valued function defined on the base space of the fiber bundle, and ∇ the covariant derivative. Then, one has

$$\nabla(\phi U) = (\nabla(\phi))U + \phi \otimes dU$$

Here, we shall use the analogy between the sections in fiber bundles and abstract moduli in order to construct the *connections* in noncommutative geometries.

6.1. Preliminaries

Let M be a right A-module for some associative *-algebra A with unit and $\mathrm{Aut}(M)$ the group of all module automorphisms of M.

Recall that M is a *finite projective A-module* if there is another A-module N such that the direct sum $M \oplus N$ is a free A-module of finite rank.

An element a of A is said to be *positive* if it may be written as a finite sum:

$$a = \sum_i b^\star{}_i \cdot b_i \tag{201}$$

with $b_i \in A$ and * is the involution of the algebra A. The set A^+ of all positive elements of A is a convex cone that we assume to be *strict* in the sense that

$$A^+ \cap (-A^+) = \{0\} \tag{202}$$

This property is typically satisfied for *-algebras of operators in Hilbert spaces.

A *Hermitian structure* on the right A-module M is an A-valued positive-definite Hermitian form (Connes, 1980)

$$(\phi, \psi) \to h(\phi, \psi) \tag{203a}$$

i.e., a sesquilinear mapping

$$h\colon \quad M \times M \to A \tag{203b}$$

which satisfies:

(a) *Hermiticity:*

$$h(\phi a, \psi b) = a^\star \cdot h(\phi, \psi) \cdot b \tag{203c}$$

$\forall \phi, \psi \in M$ and $\forall a, b \in A$.

(b) *Positive-definiteness:*

$$h(\phi, \phi) \in A^+, \qquad \forall \phi \in M \tag{203d}$$

$$h(\phi, \phi) = 0 \Rightarrow \phi = 0 \tag{203e}$$

A right A-module M equipped with a Hermitian structure h will be called a *Hermitian* A-module and the group of all A-module automorphisms U of M which preserves h, i.e.,

$$h(\phi U, \psi U) = h(\phi, \psi) \tag{204}$$

for any $\phi, \psi \in M$ is denoted by Aut(M, h).

6.2. Gauges and Gauge Transformations

Let A be an associative *-algebra with unit $\mathbb{1}$. Then, A^p is naturally a right A-module, i.e.,

$$(a_1, \ldots, a_p)\cdot a = (a_1 \cdot a, \ldots, a_p \cdot a) \tag{205}$$

$\forall (a_1, \ldots, a_p) \in A^p$ and $\forall a \in A$. It is a Hermitian A-module if one defines its Hermitian structure by

$$h((a_1, \ldots, a_p), (b_1, \ldots, b_p)) = \sum_{i=1}^{p} a_i^\star \cdot b_i \tag{206}$$

Conversely, let H^p be a free Hermitian A-module of finite rank p with Hermitian structure h. Then, one may construct an *orthonormal basis* $\{\epsilon_i\}$, $i = 1, 2, \ldots, p$, of H^p as

$$h(\epsilon_i, \epsilon_j) = \delta_{ij}\mathbb{1} \tag{207}$$

for

$$\epsilon_k = (0, \ldots, \overset{k\text{th term}}{1}, \ldots, 0) \in H^p \qquad \text{and} \qquad \forall i, j = 1, \ldots, p$$

Such a basis is called a *gauge*.

Given such a gauge, an element ϕ of H^p can be uniquely written as

$$\phi = \sum_{i=1}^{p} \epsilon_i a_i \tag{208}$$

with $a_i \in A$. Furthermore, if

$$\psi = \sum_{j=1}^{p} \epsilon_j b_j$$

is another element of H^p then [see equation (206)]

$$h(\phi, \psi) = \sum_{i=1}^{p} a_i^{\star} \cdot b_i \tag{209}$$

Thus, each gauge $\{\epsilon_i\}$ defines an isomorphism

$$H^p \to A^p \tag{210}$$

of Hermitian A-modules.

A change of orthonormal basis

$$U\colon \quad \{\epsilon_i\} \to \{\xi_i\} \tag{211a}$$

such that

$$\xi_i = \epsilon_j U_i^j \tag{211b}$$

will be naturally called a *gauge transformation.*

Such a gauge transformation is an element of

$$\mathbf{M}_p(A) = A \otimes \mathbf{M}_p(C) \tag{211c}$$

and the set of all of these *unitary* elements forms a group G_p: *The group of gauge transformations.*

Moreover, it is easy to deduce from equation (211b) that any orthonormal basis, or gauge, of A^p is of the form ϵU for a unique $U \in G_p$.

In general, if (M, h) denotes a Hermitian A-module and $\mathrm{Aut}(M, h)$ the group of all A-module automorphisms U of M which preserves the Hermitian structure h [see equations (203) and (204)], then U is called a *gauge transformation* and $\mathrm{Aut}(M, h)$ the *gauge group.*

6.3. Connections and Associated Curvatures

We define a *connection* on a right A-module M to be a linear mapping

$$\nabla\colon \quad M \to M \otimes_A \Omega^1_{\mathrm{Der}}(A) \tag{212a}$$

which satisfies the Leibniz rule

$$\nabla(\phi a) = (\nabla(\phi))a + \phi \otimes da \tag{212b}$$

for any $\phi \in M$, $a \in A$, and d the differential of the graded differential algebra $\Omega_{\mathrm{Der}}(A)$. This connection is a Ω_{D}-connection in the sense of Connes (1986).

We define a *Hermitian connection* on a Hermitian A-module M to be a connection ∇ on M which satisfies

$$d[h(\phi, \psi)] = h[\nabla(\phi), \psi) + h(\phi, \nabla(\psi)) \tag{213}$$

$\forall \phi, \psi \in M$ and h the Hermitian structure on M.

Let us remark that the difference $\nabla - \nabla'$ of Ω_D-connections on M is also a homomorphism of right A-modules in view of Born *et al.* (1926), and that (Hermitian) connections always exist on a finite projective (Hermitian) A-module M (Connes, 1986).

Following Connes (1986), one may extend the connection ∇ into a linear mapping of $M \otimes_A \Omega_{\text{Der}}(A)$ into itself:

$$\nabla: \quad M \otimes_A \Omega_{\text{Der}}(A) \rightarrow M \otimes_A \Omega_{\text{Der}}(A)$$

by setting

$$\nabla(\phi \otimes \alpha) = (\nabla(\phi))\alpha + \phi \otimes d\alpha \tag{214}$$

for any $\phi \in M$ and $\alpha \in \Omega_{\text{Der}}(A)$.

If we consider the linear mapping

$$\nabla^2: \quad M \rightarrow M \otimes \Omega^2_{\text{Der}}(A) \tag{215a}$$

then we have

$$\nabla^2(\phi a) = (\nabla^2(\phi))a \tag{215b}$$

for any $\phi \in M$ and $a \in A$. Thus, ∇^2 is a right A-module homomorphism which will represent the *curvature* associated with the connection ∇.

Let us now consider, as explained in Section 6.2, a free Hermitian right A-module of finite rank A^p equipped with a canonical basis denoted by

$$\epsilon = (\epsilon_1, \ldots, \epsilon_p) \tag{216}$$

and let ∇ be a connection on A^p. Then (see equation (212b)]

$$\nabla(\epsilon_i) = \epsilon_j \otimes \alpha^j_i \tag{217a}$$

for $i = 1, \ldots, p$ and $\alpha^j_i \in \Omega^1_{\text{Der}}(A)$, or, in a compact way,

$$\nabla(\epsilon) = \epsilon \otimes \alpha \tag{217b}$$

where $\alpha = (\alpha^j_i) \in M_p(\Omega^1_{\text{Der}}(A)) = \Omega^1_{\text{Der}}(A) \otimes \mathbf{M}_p(C))$, i.e., α is a $p \times p$ matrix whose entries are differential 1-forms.

Furthermore, ∇ is Hermitian if and only if [see equation (213)]

$$(\overline{\alpha}^j_i) = -\alpha^i_j \tag{218a}$$

or

$$\alpha^\star = -\alpha \tag{218b}$$

The quantity $\alpha \in \mathbf{M}_p(\Omega^1_{\text{Der}}(A))$ will be called the *component* of ∇ in the *gauge* ϵ [equation (216)]. Conversely, each element $\alpha \in \mathbf{M}_p(\Omega^1_{\text{Der}}(A))$ will represent the component in a gauge ϵ of a unique connection ∇.

Similarly, one can define the component of a connection ∇ in an arbitrary gauge ξ [see equation (211b)]

$$\xi = \epsilon U \tag{219}$$

It is given by

$$\nabla(\xi) = \xi \otimes \beta \tag{220a}$$

with

$$\beta = U^{-1}\alpha U + U^{-1}dU \tag{220b}$$

where α is the component of ∇ in the gauge ϵ.

Here, one may consider the component β as a component in ϵ of another (unique) connection denoted ∇^U, i.e.,

$$\nabla^U(\epsilon) = \epsilon \otimes \beta \tag{221a}$$

The mapping

$$\nabla \to \nabla^U \tag{221b}$$

with $U \in G_p$ is a *right action* of the group G_p of gauge transformations on the space of connections on A^p.

The connection ∇^U is Hermitian if and only if ∇ is Hermitian. The set

$$\{\nabla^U / U \in G_p\}$$

will be called the *gauge orbit* of ∇.

In general, if M is a right A-module, then the group Aut(M) acts on the affine space of all connections on M via a mapping (221b) with

$$\nabla^U(\phi) = U^{-1}\nabla(\phi U) \tag{222}$$

for $\phi \in M$ and $U \in$ Aut(M). With the same formula, the group Aut(M, h) acts on the space of Hermitian connections.

In the same way, the curvature ∇^2 is expressed in the gauge ϵ as

$$\nabla^2(\epsilon) = \epsilon \otimes \Omega \tag{223a}$$

where the component Ω of the curvature ∇^2 in the gauge ϵ is given by

$$\Omega = d\alpha + \alpha \wedge \alpha \in \mathbf{M}_p(\Omega^2_{\text{Der}}(A)) = \Omega^2_{\text{Der}}(A) \otimes \mathbf{M}_p(C) \tag{223b}$$

and the operator d acting on the algebra $\Omega_{\text{Der}}(A) \otimes \mathbf{M}_p(C)$ is defined by

$$d(\alpha \otimes E) = d\alpha \otimes E \tag{224}$$

$\forall\alpha \in \Omega_{\text{Der}}(A)$ and $\forall E \in \mathbf{M}_p(C)$.

If Ω is the component in a gauge ϵ of the curvature ∇^2, then the component in ϵ of the curvature $(\nabla^U)^2$ corresponding to the connection ∇^U is $\Theta = U^{-1}\Omega U$, i.e.,

$$(\nabla^U)^2(\epsilon) = \epsilon \otimes \Theta = \epsilon \otimes (U^{-1}\Omega U) \tag{225}$$

Let us now discuss the case of *flat Hermitian connections* on a free Hermitian A-module A^p. In general, a connection is called a flat connection if its associated curvature vanishes. Thus, a Ω_D-connection ∇ on A^p with component α in a gauge ϵ is flat if and only if

$$\Omega = d\alpha + \alpha \wedge \alpha = 0 \tag{226}$$

If $U \in G_p$, then ∇^U is *flat* if and only if ∇ is flat. For each gauge

$$\xi = \epsilon U^{-1} \tag{227a}$$

with $U \in G_p$, there is a *unique* connection ∇_ξ such that

$$\nabla_\xi(\xi) = 0 \tag{227b}$$

Its component in the gauge ϵ is $U^{-1}dU$, so one has

$$\nabla_\xi = \nabla_\epsilon^U \tag{227c}$$

and it is a flat Hermitian connection. These connections ∇_ϵ^U, $U \in G_p$, will be called *pure gauge connections*. The set of pure gauge connections is a gauge orbit of flat Hermitian connection on A^p.

In general, if M is a right (Hermitian) A-module and Aut(M) [Aut(M, h)] its associated gauge group, then the set of (Hermitian) connections ∇ with zero curvature, i.e., $\nabla^2 = 0$, is invariant by Aut(M)[Aut(M, h)].

7. EXAMPLES

7.1. The Limit Case $A = A_0$

For a review of the usual commutative case see, for instance, Kobayashi and Nomizu (1963), Poor (1981), and Choquet-Bruhat and de Witt-Morette (1982). Here, we just make the remark that a finite projective (Hermitian) $C^\infty(M)$-module M is the module of smooth sections of a smooth (Hermitian) vector bundle over M and a (Hermitian) connection on such a module is a (Hermitian) connection on the corresponding (Hermitian) vector bundle in the usual sense.

Moreover, if M is simply connected, then there is at most one orbit of flat Hermitian connections on a finite projective A_0-module.

These statements are not generally true in the case of noncommutative algebras A. Here, we will treat in some detail the examples of $\mathbf{M}_n(C)$ and A_0

$\otimes$ $\mathbf{M}_n(C)$ following the general procedure described below. We also give some physical interpretations and present new models of gauge theory proposed by Dubois-Violette *et al.*

For this purpose, let us recall here the definition of the $U(n)$-Yang–Mills action and in particular the Maxwell [$U(1)$-] action.

The Maxwell action is an action for connections on a $U(1)$-principal bundle over $\mathbf{R}^m$. It is also an action for Hermitian connections on a Hermitian vector bundle of rank 1 over $\mathbf{R}^m$. Finally, since $\mathbf{R}^m$ is contractile, it is an action for Hermitian connections on the free Hermitian finite projective $C^\infty(\mathbf{R}^m)$-module of rank 1. Let ∇ be such a connection with component

$$\alpha = A(x) = A_\mu(x)\, dx^\mu \in \Omega^1_{\mathrm{Der}}(C^\infty(\mathbf{R}^m)) = \Omega^1(\mathbf{R}^m)$$

(the so-called Maxwell potential), and let ∇^2 be the curvature of ∇ with component

$$\Omega = F(x) = \frac{1}{2} F_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu = d\alpha$$

(the so-called electromagnetic field), where x^μ, $\mu \in \{0, 1, \ldots, m-1\}$, are the canonical coordinates of the m-dimensional Euclidean space-time $\mathbf{R}^m$ equipped with the metric

$$ds^2 = \sum (dx^\mu)^2$$

Then, we have

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

and the Maxwell action $S(\nabla)$ for ∇ is given by

$$S(\nabla) = \|\nabla^2\|^2 = -\frac{1}{4}\int \mathrm{Tr}[F_{\mu\nu}(x)F^{\mu\nu}(x)]\, d^m x$$

This action is gauge invariant, positive, and vanishes only on the orbit of pure gauge connections. Two connections in the same gauge orbit are considered physically equivalent.

Similarly, the $U(p)$-Yang–Mills action is an action for Hermitian connections on the free Hermitian $C^\infty(\mathbf{R}^m)$-module of rank p. If ∇ is such a connection with component

$$\alpha = A_\mu(x)\, dx^\mu \in \Omega^1(\mathbf{R}^m) \otimes \mathbf{M}_p(C)$$

and ∇^2 its curvature with component

$$\begin{aligned}\Omega &= F(x)\\ &= D\alpha\\ &= d\alpha + \alpha \wedge \alpha\\ &= \frac{1}{2}\{\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)]\}\, dx^\mu \wedge dx^\nu\end{aligned}$$

then the $U(p)$-Yang–Mills action $S(\nabla)$ for ∇ is given by

$$S(\nabla) = \|\nabla^2\|^2 = -\frac{1}{4p} \int \mathrm{Tr}[F_{\mu\nu}(x)F^{\mu\nu}(x)]\, d^m x$$

which is again gauge invariant, positive, and vanishes only if ∇ is a pure gauge connection. For $p = 1$, we recover the Maxwell action.

7.2. The Case of $A = \mathbf{M}_n(C)$

Let M be a right $\mathbf{M}_n(C)$-module (Dubois-Violette *et al.*, 1990a; Dubois-Violette, 1990). A Hermitian structure h on M is an $\mathbf{M}_n(C)$-valued, positive-definite Hermitian form $h(\phi, \psi) \in \mathbf{M}_n(C)$ for $\phi, \psi \in M$, which satisfies [see equations (203)]

$$h(\phi A, \psi B) = A^\star h(\phi, \psi)B \tag{228}$$

$\forall A, B \in \mathbf{M}_n(C)$ and with $A^\star$ being the Hermitian conjugate of A, i.e.,

$$A^\star = \overline{A^t} \tag{229}$$

The pair (M, h) will be called a *Hermitian* $\mathbf{M}_n(C)$*-module.*

Since connections always exist on projective modules of finite rank, we shall restrict our considerations to the simplest Hermitian $\mathbf{M}_n(C)$-module, namely the free Hermitian $\mathbf{M}_n(C)$-module of rank 1, which we denote by H.

A Hermitian Ω_D-connection on H is a linear mapping

$$\nabla: \quad H \to H \bigotimes_{\mathbf{M}_n(C)} \Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C)) \tag{230a}$$

satisfying the following two properties:

$$\nabla(\phi A) = (\nabla(\phi))A + \phi \otimes dA \tag{230b}$$

$$d[h(\phi, \psi)] = h(\nabla(\phi), \psi) + h(\phi, \nabla(\psi)) \tag{230c}$$

$\forall \phi, \psi \in H$ and $\forall A \in \mathbf{M}_n(C)$.

An element ϵ of H such that

$$h(\epsilon, \epsilon) = \mathbb{1} \tag{231a}$$

will be called a *unitary generator* of H or a *gauge*. Given such an element, we can make the following identification:

$$H = \epsilon \mathbf{M}_n(C) \tag{231b}$$

and we also have

$$h(\epsilon A, \epsilon B) = A^\star B \tag{231c}$$

$\forall A, B \in \mathbf{M}_n(C)$.

Now, if $U \in \mathbf{M}_n(C)$ is a *unitary matrix*, then the transformation

$$U \rightarrow \epsilon U \tag{232}$$

is a bijection from the unitary group $U(n)$ onto the set of unitary generators of H. Such a change

$$\epsilon \rightarrow \xi = \epsilon U \tag{233}$$

of unitary generators is called a *gauge transformation*. The group of gauge transformations is therefore $U(n)$.

Let ∇ be a Hermitian connection on H. Given a gauge ϵ, any $\phi \in H$ may be uniquely represented as

$$\phi = \epsilon B \tag{234a}$$

with $B \in \mathbf{M}_n(C)$. Thus, in view of the definition (230b), one has

$$\nabla(\phi) = (\nabla(\epsilon))B + \epsilon \otimes dB \tag{234b}$$

where

$$\nabla(\epsilon) = \epsilon \otimes \alpha \tag{234c}$$

for a unique $\alpha \in \Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C))$ satisfying the Hermiticity condition

$$\alpha^\star = -\alpha \tag{234d}$$

in view of the Hermiticity property of ∇ [see equations (218)].

The quantities B and α in equations (234a) and (234c) will be called the components in the gauge ϵ of ϕ and ∇, respectively. Under a change of gauge [see equation (233)] induced by a unitary matrix U, these components transform as follows:

$$B \rightarrow U^{-1}B \tag{235a}$$

$$\alpha \rightarrow U^{-1}\alpha U + U^{-1}dU \tag{235b}$$

We can easily see that what is presented here is nothing other than the *noncommutative* analog of the usual notions of gauge theories. Hence, classically the quantity B corresponds to a section of some fiber bundle with a structural group to which U belongs, and α represents a local expression of the usual connection 1-form. More precisely, if we remember that $\mathbf{M}_n(C)$ is the analog of $A_0 = C^\infty(M)$ and so $U(n)$ is the analog of $U(1)$-valued functions on M, then our $\mathbf{M}_n(C)$-module H provides us with the *noncommutative* analog of electromagnetism. Namely, B and α are, respectively, the noncommutative analogs of the component of a charged scalar field and Maxwell potential in a given gauge. Nevertheless, there are some differences.

Given a gauge ϵ, there is a unique connection ∇_ϵ on H such that

$$\nabla_\epsilon(\epsilon) = 0 \tag{236a}$$

i.e.,

$$\nabla_\epsilon(\epsilon B) = \epsilon \otimes dB \tag{236b}$$

$\forall B \in \mathbf{M}_n(C)$. The component of ∇_ϵ in ϵ vanishes and its component in an arbitrary gauge $\xi = \epsilon U$ is given by

$$\alpha = U^{-1}dU \tag{236c}$$

Conversely, if the component α of a connection ∇ on H in a gauge ϵ is given by equation (236c) for some $U \in U(n)$, then one has

$$\nabla = \nabla_\xi \tag{236d}$$

with $\xi = \epsilon U^{-1}$. The connections ∇_ϵ satisfying equations (236a) and (236c) when ϵ runs over the set of unitary generators of H will be called *pure gauge connections*. They are automatically Hermitian connections on H.

As in the usual case, if ∇ and ∇' are two connections, one has

$$(\nabla - \nabla')(\phi B) = ((\nabla - \nabla')(\phi))B \tag{237a}$$

so $\nabla - \nabla'$ is a right-module homomorphism. In terms of components, this means that $\alpha - \alpha'$ transforms homogeneously:

$$\alpha - \alpha' \to U^{-1}(\alpha - \alpha')U \tag{237b}$$

under a gauge transformation induced by $U \in U(n)$:

$$\epsilon \to \epsilon U$$

where α (resp. α') is the component of ∇ (resp. ∇') in a gauge ϵ.

The set of connections on H is an affine space; however, in this case there is a natural origin ∇_0 in this space defined by the following lemma.

Lemma 3. Define the linear mapping

$$\nabla_0: \quad H \to H \bigotimes_{\mathbf{M}_n(C)} \Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C))$$

by

$$\nabla_0(\phi) = \phi \otimes (-i\theta) \tag{238a}$$

$\forall \phi \in H$ and where θ is the canonical invariant element of $\Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C))$ defined in Section 3.2.2. Then, ∇_0 is a *Hermitian connection* on H which is *gauge invariant* in the sense that

$$\nabla_0(\epsilon U) = \nabla_0(\epsilon) \tag{238b}$$

$\forall U \in U(n)$. In fact,

$$\nabla_0(\epsilon) = \epsilon \otimes (-i\theta) \tag{238c}$$

and

$$U^{-1}(-i\theta)U + U^{-1}dU = -i\theta \tag{238d}$$

$\forall U \in U(n)$. Let ∇ be a gauge-invariant connection on H, then one has

$$\nabla(\phi) = \phi \otimes [-i(\theta + \lambda_k\theta^k)] \tag{238e}$$

for some $\lambda_k \in C$, $k \in I = \{1, 2, \ldots, n^2 - 1\}$. Furthermore, ∇ is Hermitian if and only if $\lambda_k \in \mathbf{R}$, $\forall k \in I$.

Proof. In terms of $\theta = E_k\theta^k$, one has for any $B \in \mathbf{M}_n(C)$ [see equation (106a)]

$$dB = i[\theta, B] \tag{239}$$

This implies that

$$\nabla_0(\phi B) = (\nabla_0(\phi))B + \phi \otimes dB \tag{240}$$

so ∇_0 is a connection, in view of equation (212b), which is obviously Hermitian because $(-i\theta)^\star = -(-i\theta)$, in view of equation (218b).

Its component in any gauge is, by definition, $-i\theta$ and one directly verifies $U^{-1}(-i\theta)U + U^{-1}dU = -i\theta$ by using equation (239). Let ∇ be any connection on H and let $-i(\theta + \beta) \in \Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C))$ be its component in a gauge ϵ. Then, its component in a gauge ϵU, $U \in U(n)$, is $-i(\theta + U^{-1}\beta U)$. So, ∇ is gauge invariant if and only if $\beta = U^{-1}\beta U$, $\forall U \in U(n)$, which implies that $\beta = \lambda_k\theta^k$, with $\lambda_k \in C$, $\forall k \in I$. Finally, ∇ is Hermitian $\Leftrightarrow$ $\beta^\star = \beta$, i.e., $\lambda_k \in \mathbf{R}$, $\forall k \in I$. ■

Finally, let us first remark that the connection ∇_0, which will be called the canonical connection on H, cannot be a *pure gauge connection*, i.e., there is no $U \in U(n)$ such that $-i\theta = U^{-1}dU$, since it is *gauge invariant* (see Lemma 3).

Second, ∇_0 is gauge invariant and Hermitian, but it is not unique under these considerations, since these properties are also true if one replaces θ by $\theta + \lambda_k\theta^k$ for $\lambda_k \in \mathbf{R}$. However, ∇_0 is completely determined by the fact that it is the only flat connection which is not a pure gauge connection.

Let us now define the curvature associated to a connection ∇ on H. One extends the action of ∇ defined above [see equation (230a)] such that

$$\nabla\colon\ H \underset{\mathbf{M}_n(C)}{\otimes} \Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) \to H \underset{\mathbf{M}_n(C)}{\otimes} \Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) \tag{241a}$$

by setting

$$\nabla(\phi \otimes \alpha) = (\nabla(\phi))\alpha + \phi \otimes d\alpha \tag{241b}$$

$\forall\phi \in H$ and $\forall\alpha \in \Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$. Then, curvature of ∇ is the mapping

$$\nabla^2: \quad H \to H \bigotimes_{\mathbf{M}_n(C)} \Omega^2_{\mathrm{Der}}(\mathbf{M}_n(C)) \tag{242a}$$

such that ∇^2 is a right $\mathbf{M}_n(C)$-module homomorphism [see equation (215b)], i.e.,

$$\nabla^2(\phi B) = (\nabla^2(\phi))B \tag{242b}$$

$\forall\phi \in H$ and $\forall B \in \mathbf{M}_n(C)$).

In a given gauge ϵ, the component Ω of ∇^2 is defined by

$$\nabla^2(\epsilon) = \epsilon \otimes \Omega = \epsilon \otimes (d\alpha + \alpha \wedge \alpha) \tag{242c}$$

where α is the component of the connection ∇ in the gauge ϵ.

Under a *gauge* transformation $\epsilon \to \xi = \epsilon U$, $U \in U(n)$, the component Ω transforms *homogeneously*:

$$\Omega \to U^{-1}\Omega U \tag{243}$$

The pure gauge connections are *flat connections*, i.e., with zero curvature. Indeed, if [see equation (236a)]

$$\nabla_\epsilon(\epsilon) = 0$$

then [see equation (236b)]

$$\nabla_\epsilon(\epsilon B) = \epsilon \otimes dB$$

so, in view of equation (241b),

$$\nabla^2_\epsilon(\epsilon B) = \nabla_\epsilon(\epsilon) dB + \epsilon \otimes d^2 B = 0 \tag{244}$$

$\forall B \in \mathbf{M}_n(C)$.

This result is well known from the classical case $A = A_0$. However, in the noncommutative case $A = \mathbf{M}_n(C)$, flat connections which are not pure gauge connections exist. Indeed, as remarked above (see the remarks after the proof of Lemma 3), the canonical connection is the only gauge-invariant flat connection which is a pure gauge connection.

In fact, if ∇ is a gauge-invariant connection, then [see equation (238c)]

$$\nabla(\phi) = \phi \otimes [-i(\theta + \lambda_k \theta^k)]$$

for $\lambda_k \in C$, in view of Lemma 3. So, the flat connection condition

$$\nabla^2(\phi) = \phi \otimes [-id(\lambda_k\theta^k)] = \phi \otimes [iC^k_{lm}\lambda_k\theta^l \wedge \theta^m] = 0 \qquad (245a)$$

is equivalent to

$$\lambda_k = 0, \qquad \forall k \in I \qquad (245b)$$

and so, in view of equation (238a),

$$\nabla \equiv \nabla_0 \qquad (245c)$$

On the other hand, equation (106b) is equivalent to

$$\nabla_0^2 = 0 \qquad (246)$$

Thus, ∇_0 is a flat connection and it is the only gauge-invariant connection which is a flat one. We obtain the following result.

Proposition 7. Let ∇ be a Hermitian connection on H with vanishing curvature, i.e., $\nabla^2 = 0$. Then, either ∇ is a pure gauge connection ∇_ϵ or ∇ is the canonical connection ∇_0.

Proof. Choose a gauge ϵ. Then, $\nabla(\epsilon) = \epsilon \otimes \alpha$ with $\alpha^\star = -\alpha$ and $\nabla^2(\epsilon) = \epsilon \otimes \Omega$ with $\Omega = d\alpha + \alpha \wedge \alpha$. Let us choose the representation in which

$$\alpha = \beta - i\theta \qquad (247a)$$

with $\beta = B_k\theta^k$, $B_k \in \mathbf{M}_n(C)$, i.e., we choose the 1-form β which measures, in some sense, the deviation of α from the canonical connection $-i\theta$. Now, the expression for Ω reads

$$\Omega = d\alpha + \alpha \wedge \alpha = d\beta + \beta \wedge \beta - i(\theta \wedge \beta + \beta \wedge \theta) = \frac{1}{2} F_{km}\theta^k \wedge \theta^m \qquad (247b)$$

where

$$F_{km} = [B_k, B_m] - C^l_{km}B_l \qquad (247c)$$

Therefore

$$\Omega = 0 \Leftrightarrow [B_k, B_m] = C^l_{km}B_l \qquad (247d)$$

Thus, either $B_k = 0$ or $B_k = iE_k$ [because it satisfies the commutation relations of a basis of $su(n)$; see equation (83a)]. The first solution corresponds to a trivial representation of $su(n)$, which means that ∇ is the canonical connection ∇_0, and the second corresponds to a representation of $su(n)$ that is unitarily equivalent to the fundamental representation of $su(n)$, i.e.,

$$B_k = U^{-1}(iE_k)U \qquad (247e)$$

$\forall k \in I$, for some $U \in U(n)$, which reads

$$\alpha = U^{-1}(i\theta)U - i\theta = U^{-1}dU \tag{247f}$$

which means that ∇ is a pure gauge connection ∇_ξ, with $\xi = \epsilon U^{-1}$. ∎

The presence of these two distinct gauge orbits of vanishing curvature is something new as compared with the usual gauge theories.

Since our $\mathbf{M}_n(C)$-module H provides us with the analog of electromagnetism as remarked above, it is natural to introduce the analog of the Maxwell action as follows. Let ∇ be a Hermitian connection on H, α its component in a gauge ϵ, and $\Omega = d\alpha + \alpha \wedge \alpha$ the component of the associated curvature ∇^2 in ϵ with $\Omega^\star = -\Omega$. The expression

$$\langle \Omega | \Omega \rangle = \int \Omega^\star \wedge (\star\Omega) = -\int \Omega \wedge (\star\Omega) =: \|\nabla^2\|^2 \tag{248a}$$

is independent of the choice of ϵ.

Using equations (247b) and (247c), we get

$$\begin{aligned} S = \|\nabla^2\|^2 &= \frac{1}{4n} \sum_{k,m} \mathrm{Tr}[F_{km} \cdot F^{km}] \\ &= \frac{1}{4n} \sum_{k,m} \mathrm{Tr}\{([B_k, B_m] - C^l_{km}B_l)^2\} \\ &= \frac{1}{4n} \sum_{k,m} \mathrm{Tr}\{([B_k, B_m] - C^l_{km}B_l)([B^k, B^m] - C^{km}_j B^j)\} \end{aligned} \tag{248b}$$

Then, $\|\nabla^2\|^2 \geq 0$ and its absolute minima correspond to $\nabla^2 = 0$ and consist of two distinct gauge orbits: the pure gauge connections and the canonical connection ∇_0, which is a singular gauge orbit reduced to a point.

The notation used here is clearly justified since $S = \|\nabla^2\|^2$ represents the purely noncommutative analog of the classical action of the electromagnetism.

Finally, we remark that in the moduli of higher rank, the number of such minima is higher than two, as is the case for our free Hermitian $\mathbf{M}_n(C)$-module H of rank 1, and grows rapidly with the rank of the considered $\mathbf{M}_n(C)$-module M.

Up to now, we have discussed the case of H. Let us now discuss the case of an arbitrary right $\mathbf{M}_n(C)$-module M. Following Section 6.2, each gauge $\{\epsilon_i\}$ gives an isomorphism of Hermitian right finite projective A-modules [see equation (210)]. Here, $A = \mathbf{M}_n(C)$, so the only way to build a finite projective right $\mathbf{M}_n(C)$-module is to consider a tower of C^n, i.e., the space $\mathbf{M}_{Kn}(C)$ of $K \times n$ matrices with a right action of $\mathbf{M}_n(C)$. In fact, in this case, one has

$$\mathrm{Aut}(\mathbf{M}_{Kn}(C)) = GL(K)$$

with left matrix multiplication.

The module $\mathbf{M}_{Kn}(C)$ is naturally equipped with a Hermitian structure:

$$h(\phi, \psi) = \phi^{\star}\psi \tag{249}$$

where $\phi^{\star}$ is the $n \times K$ matrix Hermitian conjugate to ϕ. The gauge group is then the unitary group $U(K) \subset GL(K)$.

In this case, there is also a natural origin ∇_0 in the space of connections given by (see Lemma 3)

$$\nabla_0(\phi) = \phi \otimes (-i\theta) \tag{250a}$$

and (250b)

$$\nabla_0(\phi B) = \nabla_0(\phi)B + \phi \otimes dB$$

with $\phi \in \mathbf{M}_{Kn}(C)$, $B \in \mathbf{M}_K(C)$ and where θ is the canonical invariant element of $\Omega^1_{\mathrm{Der}}(\mathbf{M}_n(C))$.

This connection is Hermitian and it follows from equation (106b) that its curvature vanishes [see equation (246)]:

$$\nabla_0^2(\phi) = \phi \otimes [d(-i\theta) + (-i\theta)^2] = 0 \tag{250c}$$

Any connection ∇ is of the form

$$\nabla(\phi) = \nabla_0(\phi) + \phi \otimes A = \nabla_0(\phi) + A_k\phi \otimes \theta^k \tag{250d}$$

where $A = A_k\theta^k$, with $A_k \in \mathbf{M}_K(C)$.

The connection ∇ is Hermitian if and only if the A_k are anti-Hermitian:

$$A_k^{\star} = -A_k \tag{250e}$$

The curvature of ∇ is given by

$$\nabla^2(\phi) = \phi \otimes F = \frac{1}{2} F_{kl}\phi \otimes \theta^k \wedge \theta^l \tag{250f}$$

with

$$F = \frac{1}{2} F_{kl}\theta^k \wedge \theta^l = \frac{1}{2} ([A_k, A_l] - C^m_{kl}A_m)\theta^k \wedge \theta^l \tag{250g}$$

$\forall\phi \in \mathbf{M}_{Kn}(C)$ and $A_k \in \mathbf{M}_K(C)$. Here, the symbols A and F are used to parallel the usual notations in gauge theories.

Thus, $\nabla^2 = 0$ if and only if the A_k form a representation of $su(n)$ (or $sl(n)$) in C^K. Two such connections are in the same orbit of $\mathrm{Aut}(\mathbf{M}_{Kn}(C))$ if and only if the corresponding representations of $su(n)$ are *equivalent*. This implies that the gauge orbits of flat Hermitian connections are in one-to-one correspondence with the unitary classes of representations of $su(n)$ in C^K. For instance, if $n = 2$, these orbits are labeled by the number of partitions of the integer K.

7.3. The Case of $A = A_0 \otimes \mathbf{M}_n(C)$

7.3.1. The Analogs of the Euclidean Maxwell and Yang–Mills Theories

From the isomorphism (210) of Hermitian right finite projective A-modules of rank p, with $A = C^\infty(M) \otimes \mathbf{M}_n(C)$, a gauge transformation U is a unitary element of $\mathbf{M}_p(A) = A \otimes \mathbf{M}_p(C) = C^\infty(M) \otimes \mathbf{M}_{pn}(C)$. So, U is a $U(np)$-valued function on the m-dimensional manifold M (Kerner, 1990; Dubois-Violette *et al.*, 1990b; Dubois-Violette, 1990).

On the other hand, we know that in the commutative case where $A = A_0$, the pure gauge connections are the only flat Hermitian connections on A_0^p, and we have shown in Section 7.2 that in the case of $A = \mathbf{M}_n(C)$, the orbits of flat Hermitian connections are in one-to-one correspondence with the unitary classes of representations of $su(n)$.

Now, we will study the gauge orbits of Hermitian flat connections on a right Hermitian finite projective A-module A^p, where $A = A_0 \otimes \mathbf{M}_n(C)$, with $n \geq 2$ and $p \geq 1$.

To construct *Hermitian* connections on A^p, it is necessary and sufficient to construct *anti-Hermitian* components in a canonical basis ϵ of A^p [in view of equation (218b)]. Then, let R_k^η, $k \in I = \{1, \ldots, n^2 - 1\}$, $\eta \in J = \{0, 1, \ldots, N(n, p)\}$, be a set of *anti-Hermitian* elements of $\mathbf{M}_n(C) \otimes \mathbf{M}_p(C) = \mathbf{M}_{np}(C)$ such that

$$R_k^0 = 0 \tag{251a}$$

$$R_k^1 = iE_k \otimes 1 \tag{251b}$$

$$[R_k^\eta, R_l^\eta] = C_{kl}^m R_m^\eta \tag{251c}$$

$\forall \eta \in J$ and $k, l \in I$ [i.e., R^η is a representation of $su(n)$ in $C^n \otimes C^p$], and such that, if $\{R_k\}$ are $n^2 - 1$ anti-Hermitian elements of $\mathbf{M}_{np}(C)$ satisfying

$$[R_k, R_l] = C_{kl}^m R_m \tag{252}$$

$\forall k, l \in I$, then there is a unique $\eta \in J$ and a unitary $V \in \mathbf{M}_{np}(C)$ such that

$$R_k = V^{-1} R_k^\eta V \tag{253}$$

$\forall k \in I$. This means that the set $\{R^\eta\}$, $\eta \in J$, is a complete set of mutually inequivalent anti-Hermitian representations of $su(n)$ in $C^n \otimes C^p$.

Then, the connections ∇ on A^p with components

$$\alpha = (R_k^\eta - iE_k \otimes 1)\theta^k \in \mathbf{M}_p(\Omega^1_{\mathrm{Der}}(A)) = \Omega^1_{\mathrm{Der}}(A) \otimes \mathbf{M}_p(C) \quad (254)$$

in a gauge ϵ are *Hermitian* and we will denote them by $\overset{\eta}{\nabla}$.

The following theorem results.

Theorem 1. (a) The connections $\overset{\eta}{\nabla}$ are flat Hermitian connections and, if $\eta \neq \zeta$, the gauge orbits of $\overset{\eta}{\nabla}$ and of $\overset{\zeta}{\nabla}$ are distinct.

(b) A Hermitian connection ∇ on A^p is flat if and only if it is an element of the gauge orbit of $\overset{\eta}{\nabla}$ for some $\eta \in J$, i.e., $\nabla \equiv \overset{\eta}{\nabla}{}^U$ with $U \in G_p$ and $\eta \in J$.

Proof. (a) Let ∇ be a Hermitian connection on A^p with a component ω in a given gauge ϵ. Let ω be given by

$$\omega = \beta + (B_k - iE_k \otimes 1)\theta^k \quad (255)$$

where β is a 1-form on M with values in the anti-Hermitian elements of $\mathbf{M}_{np}(C)$ and where B_k are functions on M with values in the anti-Hermitian elements of $\mathbf{M}_{np}(C)$. Then, one has

$$\Omega = d\omega + \omega \wedge \omega$$
$$= d_1\beta + \beta \wedge \beta + (d_1B_k + [\beta, B_k])\theta^k + \frac{1}{2}\{[B_k, B_l] - C_{kl}^m B_m\}\theta^k \wedge \theta^l \quad (256)$$

Then, ∇ is flat (i.e., $\Omega = 0$) if and only if

$$d_1\beta + \beta \wedge \beta = 0 \quad (257a)$$

$$d_1B_k + [\beta, B_k] = 0 \quad (257b)$$

and (257c)

$$[B_k, B_l] = C_{kl}^m B_m$$

$\forall k, l \in I$ and where d_1 is the unique antiderivation of $\Omega_{\mathrm{Der}}(A)$ extending the exterior differential of $\Omega(M)$ such that $d_1(\Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) = 0$. It follows that every connection ∇ on A^p with component ω given by equation (255) and satisfying equations (257) is a flat Hermitian connection on A^p. In particular, the connections $\overset{\eta}{\nabla}$ are flat connections.

If

$$\overset{\zeta}{\nabla} = \overset{\eta}{\nabla}{}^U = U^{-1}\overset{\eta}{\nabla} U, \qquad \text{for} \quad \zeta \neq \eta$$

the unitary element U of $\mathbf{M}_p(A) = A_0 \otimes \mathbf{M}_{np}(C)$ may be chosen to be constant and then

$$R_k^{\zeta} = U^{-1} R_k^{\eta} U$$

but this contradicts the statement that $\{R^{\eta}\}$, $\eta \in J$, is a complete set of mutually inequivalent anti-Hermitian representations of $su(n)$ in $C^n \otimes C^p$. Then, if $\zeta \neq \eta$, the gauge orbits of the two flat Hermitian connections $\overset{\eta}{\nabla}$ and $\overset{\zeta}{\nabla}$ are distinct.

(b) Assume that ∇ is a flat Hermitian connection. Then, the relation (257a) implies that β is a pure gauge connection, i.e.,

$$\beta = U^{-1} d_1 U$$

and the relation (257c) means that [see equation (251c)]

$$B_k = V^{-1} R_k^{\eta} V$$

for some $\eta \in J$ and U, $V \in G_p$. Furthermore, equation (257b) implies that

$$d_1(UV^{-1} R_k^{\eta} V U^{-1}) = 0$$

So we can choose U and V such that $U = V$. This implies that $\nabla \equiv \overset{\eta}{\nabla}{}^U$. ■

Finally, let us make the following remarks.

First, under a gauge transformation

$$\nabla \to \nabla^U \tag{258a}$$

$U \in G_p$, the 1-form β and the functions B_k defined above transform as

$$\beta \to U^{-1} \beta U + U^{-1} d_1 U \tag{258b}$$

and (258c)

$$B_k \to U^{-1} B_k U$$

respectively. Then, the B_k transform homogeneously. In fact, this explains why the components α and ω have been chosen to be of the form given by equations (254) and (255), respectively. This is deeply tied to what is described in Lemma 3 (see Section 7.2) concerning matrix algebras.

Second, one has

$$\overset{1}{\nabla} = \nabla_{\epsilon} \tag{259}$$

So, the pure gauge connections on A^p are the elements of the gauge orbit of $\overset{1}{\nabla}$.

The third remark is relative to the classification of gauge orbits of flat Hermitian connections on A^p. One has:

1. For $p \geq 1$, $n \geq 2$, $N(n, p) \geq 1$, there are at least two gauge orbits, namely the orbit of $\overset{0}{\nabla}$ and the orbit of $\overset{1}{\nabla}$.
2. For $p = 1$, $n \geq 2$, $N(n, 1) = 1$, there are only these two gauge orbits.

Lastly, we remark that formulas like (256) appear naturally in the double-bundle structures (Kerner *et al.*, 1987).

At this point, it is straightforward to generalize the classical Maxwell action and, in general, the $U(p)$-Yang–Mills action [see Section 7.1 for the case $n = 1$, i.e., $A = A_0 = C^\infty(\mathbf{R}^m)$] for an arbitrary positive integer $p \geq 0$ by considering the associative algebra $A = C^\infty(\mathbf{R}^m) \otimes \mathbf{M}_n(C)$ with $n \geq 2$.

Let x^μ, $\mu \in \{0, 1, \ldots, m - 1\}$, be the canonical coordinates of the m-dimensional Euclidean space-time $\mathbf{R}^m$ equipped with the metric

$$ds^2 = \sum (dx^\mu)^2 \tag{260}$$

It is natural to generalize the Euclidean Maxwell action (see Section 7.1) for an arbitrary positive integer n as $\|\nabla^2\|^2$ on the Hermitian connection ∇ on the free Hermitian A-module of rank one, and so for the $U(p)$-Yang–Mills action in the case of a free Hermitian A-module of rank p.

Let $\omega \in \Omega^1_{\mathrm{Der}}(A)$ and $\Omega \in \Omega^2_{\mathrm{Der}}(A)$ be the components of ∇ and ∇^2, respectively; then

$$\|\nabla^2\|^2 = \langle \Omega | \Omega \rangle = \langle d\omega + \omega \wedge \omega | d\omega + \omega \wedge \omega \rangle \tag{261}$$

where the scalar product on $\Omega_{\mathrm{Der}}(A)$ is defined in Section 3.3 [see equations (150)].

Writing again ω as [see equation (255) for $p = 1$]

$$\omega = \alpha_1 + (\alpha_2 - i\theta) = A_\nu(x) d_1 x^\nu + \rho(A_k(x) - iE_k)\theta^k \tag{262}$$

with anti-Hermitian $n \times n$ matrix-valued functions $A_\nu(x)$, $\nu \in \{0, 1, \ldots, m - 1\}$, and $A_k(x)$, $k \in \{1, 2, \ldots, n^2 - 1\}$, the *generalized Maxwell action* $S(\nabla)$ reads

$$S(\nabla) = \|\nabla^2\|^2 = -\frac{1}{4n}\int \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]\, d^m x - \frac{1}{2n}\int \mathrm{Tr}[F_{\mu i}F^{\mu i}]\, d^m x - \frac{1}{4n}\int \mathrm{Tr}[F_{kl}F^{kl}]\, d^m x \tag{263}$$

with ρ some positive constant of dimension of length [see equation (146)], and

$$\Omega = \frac{1}{2} F_{\mu\nu} d_1 x^\mu \wedge d_1 x^\nu + \rho F_{\mu i} d_1 x^\mu \wedge \theta^i + \frac{1}{2} \rho^2 F_{kl} \theta^k \wedge \theta^l \tag{264a}$$

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = D_\mu A_\nu \\ &= (\partial_\mu A^0_\nu - \partial_\nu A^0_\mu) \otimes \mathbb{1} + (\partial_\mu A^k_\nu - \partial_\nu A^k_\mu - iC^k_{ij} A^i_\mu A^j_\nu) \otimes E_k \end{aligned} \tag{264b}$$

$$\begin{aligned} F_{\mu i} &= \partial_\mu A_i + [A_\mu, A_i] = \nabla_\mu A_i \\ &= \partial_\mu A^0_i \otimes \mathbb{1} + (\partial_\mu A^k_i - iC^k_{rs} A^r_\mu A^s_i) \otimes E_k \end{aligned} \tag{264c}$$

$$\begin{aligned} F_{kl} &= [A_k, A_l] - \frac{1}{\rho} C^i_{kl} A_i = D_k A_l \\ &= A^0_i C^i_{kl} \otimes \mathbb{1} + \left(-iC^r_{pq} A^p_k A^q_l - \frac{1}{\rho} C^i_{kl} A^r_i \right) \otimes E_r \end{aligned} \tag{264d}$$

Under a gauge transformation $\nabla \to \nabla^U$, $U \in G_1$, the A_μ, A_i, and $F_{\mu i}$ transform, respectively, as

$$A_\mu \to U^{-1} A_\mu U + U^{-1} \partial_\mu U \tag{265a}$$

$$A_i \to U^{-1} A_i U \tag{265b}$$

$$F_{\mu i} \to U^{-1} F_{\mu i} U \tag{265c}$$

The action S given by equation (263) is gauge invariant, positive, and, for $n \geq 2$, vanishes on the gauge orbit of ($A_\mu = 0$, $A_i = 0$) and the gauge orbit of ($A_\mu = 0$, $A_k = (i/\rho) E_k$).

One can also obtain the same results by another method, as follows. One has (see Lemma 2 in Section 3.3.1)

$$\mathrm{Der}(A) = \{\mathrm{Der}[C^\infty(\mathbf{R}^m) \otimes \mathbb{1}]\} \oplus \{C^\infty(\mathbf{R}^m) \otimes \mathrm{Der}[\mathbf{M}_n(C)]\} \tag{266a}$$

i.e., all vector fields in the space Der(A) are of the form [see equation (135)]

$$\chi = \chi^\mu(x) \partial_\mu \otimes \mathbb{1} + \chi^k(x) \otimes \mathrm{ad}(iE_k) \tag{266b}$$

where $\chi^\mu(x)$, $\chi^k(x) \in C^\infty(\mathbf{R}^m)$ with $\mu \in \{0, 1, \ldots, m-1\}$, $k \in \{1, 2, \ldots, n^2 - 1\}$.

We have also [see equation (137)]

$$\Omega_{\mathrm{Der}}(A) = \Omega_{\mathrm{Der}}(C^\infty(\mathbf{R}^m)) \otimes \Omega_{\mathrm{Der}}(\mathbf{M}_n(C)) \tag{267a}$$

where the differential d of $\Omega_{\mathrm{Der}}(A)$ is given by

$$d = d_1 + \rho d_2 \tag{267b}$$

where d_1 is the differential along $\mathbf{R}^m$ and d_2 the differential of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$.

To define a connection on our module, we need to define the *covariant derivative* D on $\Omega^1_{\text{Der}}(A)$. This method consists in assuming that D reads

$$D = D_1 + \rho D_m + \rho^2 D_2 = d + \alpha \wedge \cdot \tag{267c}$$

where D_1 (respectively, D_2) is the *covariant derivative* on the *commutative* part $\Omega^1(R^m)$ [respectively, on the *noncommutative* part $\Omega^1_{\text{Der}}(\mathbf{M}_n(C))$] of $\Omega_{\text{Der}}(A)$, given by

$$D_1 = d_1 + \alpha_1 \wedge \cdot \tag{267d}$$

$$D_2 = d_2 + \alpha_2 \wedge \cdot \tag{267e}$$

where α_1 (respectively, α_2) is the *component* of the commutative part ∇_1 (respectively, of the noncommutative part ∇_2) of a connection $\nabla = \epsilon \otimes \alpha \in \Omega^1_{\text{Der}}(A)$, and D_m is an *expression* which mixes the *commutative* part with the *noncommutative* one (we will call it the *mixing* part) and which will be defined below.

The component α in some gauge ϵ of any 1-form [element of $\Omega^1_{\text{Der}}(A)$] may be written as

$$A \equiv \alpha = \alpha_1 + \rho\alpha_2 = A_\nu(x)\, d_1x^\nu + \rho A_i(x)\theta^i \tag{268a}$$

and is defined on the free Hermitian finite projective A-module $\mathbf{M}_p(A) = C^\infty(\mathbf{R}^m) \otimes \mathbf{M}_{np}(C)$. The quantities $A_\nu(x)$ and $A_i(x)$ read

$$A_\nu(x) = A^0_\nu(x) \otimes \mathbb{1} + A^k_\nu(k) \otimes E_k \tag{268b}$$

$$A_i(x) = A^0_i(x) \otimes \mathbb{1} + A^j_i(x) \otimes E_j \tag{268c}$$

where $A^0_\nu(x)$, $A^k_\nu(x)$, $A^0_i(x)$, and $A^j_i(x) \in C^\infty(\mathbf{R}^m)$.

Let

$$\nabla(\epsilon) = \epsilon \otimes \alpha \tag{269}$$

be a Hermitian connection on this module with component α given by (268).

Then, the component Ω of the associated curvature

$$\nabla^2(\epsilon) = \epsilon \otimes \Omega \tag{270}$$

is given by

$$F \equiv \Omega = D\alpha = d\alpha + \alpha \wedge \alpha = \tfrac{1}{2}F_{ab}\sigma^a \wedge \sigma^b = F_1 + \rho F_m + \rho^2 F_2 \tag{271a}$$

with $a, b, \ldots = (\mu, k), (\nu, i), \ldots;\ \mu, \nu \in \{0, 1, \ldots, m-1\}$; $k, i \in \{1, 2, \ldots, n^2 - 1\}$; and

$$\begin{aligned} F_1(x) &= \tfrac{1}{2}F_{\mu\nu}(x) d_1x^\mu \wedge d_1x^\nu \\ &= \tfrac{1}{2}[F^0_{\mu\nu}(x) \otimes \mathbb{1} + F^k_{\mu\nu}(x) \otimes E_k] d_1x^\mu \wedge d_1x^\nu \end{aligned} \tag{271b}$$

$$F_m(x) = F_{\mu i}(x) d_1 x^\mu \wedge \theta^i$$
$$= [F^0_{\mu i}(x) \otimes 1 + F^j_{\mu i}(x) \otimes E_j]\, d_1 x^\mu \wedge \theta^i \tag{271c}$$
$$F_2(x) = \tfrac{1}{2} F_{kl}(x) \theta^k \wedge \theta^l$$
$$= \tfrac{1}{2}[F^0_{kl}(x) \otimes 1 + F^m_{kl}(x) \otimes E_m] \theta^k \wedge \theta^l \tag{271d}$$

Using the relations (79)–(84), (96b), and (101) and the property

$$\sigma^a \wedge \sigma^b = -\sigma^b \wedge \sigma^a \tag{272}$$

$\forall a = (\mu, k)$ and $b = (\nu, l)$, with $\mu, \nu \in \{0, 1, \ldots, m-1\}$ and $k, l \in \{1, 2, \ldots, n^2 - 1\}$, we get

$$F^0_{\mu\nu}(x) = \partial_\mu A^0_\nu(x) - \partial_\nu A^0_\mu(x) \tag{273a}$$
$$F^k_{\mu\nu}(x) = \partial_\mu A^k_\nu(x) - \partial_\nu A^k_\mu(x) - iC^k_{ij} A^i_\mu(x) A^j_\nu(x) \tag{273b}$$
$$F^0_{\mu k}(x) = \partial_\mu A^0_k(x) \tag{273c}$$
$$F^l_{\mu k}(x) = \partial_\mu A^l_k(x) - iC^l_{rs} A^r_\mu(x) A^s_k(x) \tag{273d}$$
$$F^0_{kl}(x) = -A^0_i(x) C^i_{kl} \tag{273e}$$
$$F^m_{kl}(x) = -iC^m_{rs} A^r_k(x) A^s_l(x) - \frac{1}{\rho} C^i_{kl} A^m_i(x) \tag{273f}$$

It is easy to verify that

$$F_1(x) = D_1 \alpha_1$$
$$= d_1 \alpha_1 + \alpha_1 \wedge \alpha_1$$
$$= \tfrac{1}{2} F_{\mu\nu}(x)\, d_1 x^\mu \wedge d_1 x^\nu$$
$$= \tfrac{1}{2}\{\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\}\, d_1 x^\mu \wedge d_1 x^\nu$$
$$= \tfrac{1}{2} D_\mu A_\nu\, d_1 x^\mu \wedge d_1 x^\nu \tag{274a}$$
$$F_m(x) \equiv D_m(\alpha_1, \alpha_2)$$
$$= F_{\mu k}(x)\, d_1 x^\mu \wedge \theta^k$$
$$= \partial_\mu A_k(x)\, d_1 x^\mu \wedge \theta^k + 2\alpha_1 \wedge (\alpha_2 - i\theta)$$
$$= \{\partial_\mu A_k - [A_\mu, A_k]\}\, d_1 x^\mu \wedge \theta^k$$
$$= \nabla_\mu A_k(x)\, d_1 x^\mu \wedge \theta^k \tag{274b}$$

$$
\begin{aligned}
F_2(x) &= D_2\alpha_2 \\
&= d_2\alpha_2 + \alpha_2 \wedge \alpha_2 \\
&= \tfrac{1}{2}F_{kl}(x)\theta^k \wedge \theta^l \\
&= \frac{1}{2}\left\{[A_k, A_l] - \frac{1}{\rho}\, C^i_{kl}A_i\right\}\theta^k \wedge \theta^l \\
&= \tfrac{1}{2}D_kA_l(x)\theta^k \wedge \theta^l
\end{aligned} \tag{274c}
$$

where $\theta = E_k\theta^k$ is the *canonical invariant element* of $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$.

Then, the connection ∇ [see equation (269)] is a *flat connection*, i.e., $F \equiv 0$, if and only if [see equations (271a), (273), and (274)]:

$$F_1(x) = 0 \Leftrightarrow F_{\mu\nu}(x) = 0 \tag{275a}$$

$$F_m(x) = 0 \Leftrightarrow F_{\mu k}(x) = 0 \tag{275b}$$

$$F_2(x) = 0 \Leftrightarrow F_{kl}(x) = 0 \tag{275c}$$

Here, we meet the same results obtained by the first method that are given by equations (257a)–(257c) [see also equations (264)].

This shows the importance of the introduction of the notion of *origin* of the affine space of connections on the module, i.e., the *canonical connection* ∇_0 (see Lemma 3 in Section 7.2).

One can generalize similarly the $U(p)$-Yang–Mills action by writing the action for a Hermitian connection on the free Hermitian finite projective A-module of rank p. The action has again the form given by equation (263), but now the A_μ and the A_k are $(n\ p) \times (n\ p)$ anti-Hermitian matrix-valued. Thus, using Theorem 1, there are as many gauge orbits of connections on which the action vanishes as there are unitary classes of anti-Hermitian representations of $su(n)$ in $C^n \otimes C^p$.

Finally, one can extend these results to the general case of a finite projective right $C^\infty(\mathbf{R}^m) \otimes \mathbf{M}_n(C)$-module, namely $C^\infty(\mathbf{R}^m) \otimes M_{Kn}(C)$. This module is free of rank $K\cdot n$, so $d_1\phi(x)$ is well defined for $\phi \in C^\infty(\mathbf{R}^m) \otimes \mathbf{M}_{Kn}(C)$ (Dubois-Violette, 1990):

$$d_1\phi(x) = \partial_\mu\phi(x)\, d_1x^\mu \tag{276}$$

A connection on this module is of the form

$$\nabla(\phi) = d_1\phi - i\rho\phi\theta + A\phi \tag{277a}$$

with

$$A = A_\mu(x)\, d_1x^\mu + \rho A_k(x)\theta^k \tag{277b}$$

where the A_μ and the A_k are $K \times K$-matrix-valued functions on $\mathbf{R}^m$, i.e., elements of $C^\infty(\mathbf{R}^m) \otimes \mathbf{M}_k(C)$, and where

$$A\phi(x) = A_\mu(x)\phi(x)\, d_1x^\mu + \rho A_k(x)\phi(x)\theta^k \tag{277c}$$

Such a connection is *Hermitian* if and only if the $A_\mu(x)$ and the $A_k(x)$ are *anti-Hermitian*, $\forall x \in \mathbf{R}^m$. The curvature ∇^2 of ∇ is given by

$$\nabla^2(\phi) = F\phi \tag{278a}$$

where

$$\begin{aligned} F = {}& \tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])\, d_1x^\mu \wedge d_1x^\nu \\ & + \rho(\partial_\mu A_k + [A_\mu, A_k])\, d_1x^\mu \wedge \theta^k \\ & + \tfrac{1}{2}\rho^2\left([A_k, A_l] - \frac{1}{\rho} C^m_{kl}A_m\right)\theta^k \wedge \theta^l \end{aligned} \tag{278b}$$

The connection ∇ is *flat* (i.e., $\nabla^2 = 0$) if and only if each term of F vanishes, which implies that ∇ is gauge equivalent to a connection for which one has

$$A_\mu = 0 \tag{279a}$$

$$\partial_\mu A_k = 0 \tag{279b}$$

and (279c)

$$[A_k, A_l] = \frac{1}{\rho} C^m_{kl}A_m$$

Furthermore, two such connections are *equivalent* if and only if the corresponding representation of $su(n)$ in C^k (given by the constant $K \times K$ matrices A_k) are *equivalent*. So again, the gauge orbits of flat Hermitian connections are in one-to-one correspondence with the unitary classes of (anti-Hermitian) representations of $su(n)$ in C^k.

For instance, for $n = 2$, the number of such orbits is again the number of partitions of the integer K, exactly as in the case $A = \mathbf{M}_n(C)$ (see Section 7.2).

It is clear from equation (278b) that the generalization of the Euclidean Yang–Mills action for a Hermitian connection ∇ [see equation (277a)] on $C^\infty(\mathbf{R}^m) \otimes \mathbf{M}_{Kn}(C)$ is

$$\begin{aligned} S(\nabla) = \|F\|^2 = {}& -\frac{1}{4}\int \mathrm{Tr}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])^2\, d^mx \\ & - \frac{1}{2\rho^2}\int \mathrm{Tr}(\partial_\mu A_k + [A_\mu, A_k])^2\, d^mx \\ & - \frac{1}{4\rho^4}\int \mathrm{Tr}\left([A_k, A_l] - \frac{1}{\rho} C^m_{kl}A_m\right)^2 d^mx \end{aligned} \tag{280}$$

where the metric of the space-time $\mathbf{R}^m$ is

$$g_{\mu\nu} = \delta_{\mu\nu} \tag{281a}$$

with the basis $\{1, E_k\}$ chosen in such a way that

$$K_{kl} = \delta_{kl} \tag{281b}$$

i.e. [in view of equation (80)],

$$\mathrm{Tr}(E_k E_l) = n\delta_{kl}$$

and where the scalar product $\langle , \rangle$ on $\Omega_{\mathrm{Der}}(C^\infty(\mathbf{R}^m) \otimes \mathbf{M}_n(C))$ is defined by equation (149).

Let us now discuss the physical interpretation of all these results. The action $S(\nabla)$ given by equation (280) is the Yang–Mills action on the noncommutative space corresponding to the associative algebra $A = C^\infty(\mathbf{R}^m) \otimes \mathbf{M}_n(C)$. It can be interpreted as the Euclidean action of a field theory on the m-dimensional space-time $\mathbf{R}^m$. This field theory consists of a $U(n)$-Yang–Mills potential field $A_\mu(x)$ minimally coupled to $n^2 - 1$ scalar fields $A_k(x)$ with values in the adjoint representation of $U(n)$ which interact among themselves through a quartic polynomial potential.

Let $n \geq 2$ and $m \geq 2$. Then the action (280) of this quantum field theory is positive and vanishes for the trivial gauge orbit Ω_0 defined by

$$A_\mu = 0 \qquad \text{and} \qquad A_k = 0 \tag{282}$$

but also vanishes on other gauge orbits that are labeled by unitary classes of representations of $su(n)$ in C^K. Recall that, for $p = 1$ [see equation (263)], and in addition to Ω_0, there is only *one nontrivial gauge orbit* Ω_1 defined by

$$A_\mu = 0 \qquad \text{and} \qquad A_k = \frac{i}{\rho} E_k \tag{283}$$

In this case, these two gauge orbits, namely Ω_0 and Ω_1, are separated by an infinite potential barrier, i.e., there is no instanton interpolating between them. This comes from the translation invariance.

Therefore, by standard arguments (Itzykson and Zuber (1980)), each of these orbits corresponds to a vacuum for the corresponding quantum field theory. Let us denote these vacua by the same notation, i.e., by Ω_α, $\alpha = 0$, $1, \ldots$ (for $p = 1$, $\alpha = 0, 1$).

The dynamics of the theory can be developed around one of these vacua. Then, one must choose a vacuum and use the field variables that are adapted to this vacuum, i.e., translate the fields in such a way that they must vanish up to a gauge transformation on the gauge orbit corresponding to the chosen vacuum in order that the zero (vacuum expectation values) of the associated translated quantum fields correspond to the vacuum.

Thus, the variables A_μ and A_k are adapted to the vacuum sector Ω_0 corresponding to the *trivial representation* (282). In this sector, the $U(1)$- and $SU(n)$-gauge fields A^0_μ and A^k_μ are massless, whereas the scalar fields A_k describing the excitations around $A_k = 0$ are massive with the same mass:

$$m(A_k) = \frac{n}{\rho} \tag{284}$$

The multiplet A_k contains many scalar fields (Higgs fields), because

$$A_k(x) = A^0_k(x) \otimes \mathbb{1} + A^l_k(x) \otimes E_l$$

For $p = 1$ and for the second vacuum Ω_1, the field variables that are adapted to this vacuum are A_μ and B_k, with

$$B_k = A_k - \frac{i}{\rho} E_k = A^0_k(x) \otimes \mathbb{1} + \left(A^l_k(x) - \frac{i}{\rho}\delta^l_k\right) \otimes E_l \tag{285a}$$

For the translation

$$A_k \to B_k \Leftrightarrow A^l_k(x) \to A^l_k(x) - \frac{i}{\rho}\delta^l_k \tag{285b}$$

the Abelian $U(1)$-gauge field A^0_μ remains *massless*, while the $SU(n)$-gauge field A^k_μ acquires a *nonzero* mass due to the contribution of the quartic term:

$$\frac{1}{2\rho^2}\,\mathrm{Tr}([A_\mu, E_k][A^\mu, E^k])$$

in the traceless part [i.e., the $SU(n)$ part] of the A_μ.

Then all the components A^k_μ have the same mass:

$$m(A_\mu) = \frac{(2)^{1/2} n}{\rho} \tag{286}$$

The components A^0_k of the Higgs field acquires the mass

$$m(A^0_k) = \frac{2}{\rho} \tag{287}$$

and finally the Higgs multiplet A^l_k develops quite a complicated mass spectrum. We shall describe this spectrum in the case $n = 2$ (see Section 7.3.2).

In general, for an arbitrary p, the quantum theory possesses many vacua Ω_α, $\alpha \in \{0, 1, \ldots, N(n, p)\}$, where the number $N(n, p)$ grows very quickly with p for $n \geq 2$. If one chooses a vacuum Ω_α corresponding to the representation R^{η}_k of $su(n)$ [see equation (251c)], the field variables that are adapted to this vacuum sector are A_μ and B^{η}_k, with

$$B_k^{\eta} = A_k - \frac{1}{\rho} R_k^{\eta} \tag{288}$$

Making this change of variables, one observes that the components A_μ become massive and that the B_k^{η} have different masses. The whole mass spectrum depends on α. This is very analogous to the *Higgs mechanism.*

However, the gauge invariance is not broken here, and the noninvariance of the mass terms of the A_μ is compensated by the fact that the gauge transformation of the B_k^{η} becomes *inhomogeneous* (they are components of a connection).

7.3.2. Simple Models

From the above subsection, we see that the simplest analog of the Euclidean Maxwell theory on a noncommutative space corresponding to the associative algebra $A = C^\infty(\mathbf{R}^4) \otimes \mathbf{M}_2(C)$ contains a $U(1)$-gauge field A_μ^0, an $SU(2)$-gauge field A_μ^k, a scalar triplet A_k^0, and a scalar Higgs multiplet (a 3×3 matrix of scalar fields) A_k^l (Kerner, 1990; Dubois-Violette *et al.*, 1990b; Dubois-Violette, 1990).

In this case, there are only two vacua Ω_0 and Ω_1 corresponding to the two inequivalent representations of $su(2)$ in C^2: the trivial one $\{0\} \oplus \{0\}$ and the representation $\{1/2\}$ corresponding to the spin 1/2 (we will denote these two vacua by Ω_0 and $\Omega_{1/2}$ respectively).

For the vacuum Ω_0, the variables A_μ and A_k are adapted to this vacuum and there is not much to say.

The vacuum $\Omega_{1/2}$ corresponds to the gauge orbit of pure gauge connections on the free Hermitian A-module of rank one. The vacuum sector of $\Omega_{1/2}$ is therefore very natural from the point of view of the underlying noncommutative differential geometry. Then, for the vacuum $\Omega_{1/2}$ one has to make the translation in the A_k as given by equations (285).

Let us decompose the expression $B_k^m = A_k^m - (i/\rho)\delta_k^m$ into its irreducible parts as

$$B_k^m = \tau\delta_k^m + s_k^m + a_k^m \tag{289a}$$

where the first term, with

$$\tau = \frac{1}{3} B_m^m \tag{289b}$$

represents *pure trace*, the second term (s_k^m) the *traceless symmetric* part, and the last term (a_k^m) the *traceless antisymmetric part.*

From the action (280) and using equations (285), one obtains the following mass spectrum:

The fields B_k^0 have mass [see equation (287)]

$$m_0 = m(B_k^0) = 2/\rho \tag{290a}$$

the field τ has mass

$$m_\tau = 2/\rho \tag{290b}$$

the fields s_k^m have mass

$$m_s = 4/\rho \tag{290c}$$

and the fields a_k^m are massless:

$$m_a = 0 \tag{290d}$$

Notice that, in contrast to the A_k, the B_k transform inhomogeneously under a gauge transformation and one can fix the gauge by imposing

$$a_k^m = 0 \tag{291}$$

This model with vacuum $\Omega_{1/2}$ is interesting, but not very realistic, since the obtained mass spectrum is still quite far from the Weinberg–Salam model of electroweak interactions. For instance, in this model one has

$$m_W = m_Z$$

and there is no mixing between A_μ^0 and A_μ^3 as in the standard model.

Moreover, although this model with $\Omega_{1/2}$ looks a little like the bosonic sector for the Weinberg–Salam model with the $U(1) \times SU(2)$ group, one must identify the $U(1)$-gauge potential with $\mathrm{Tr}(A_\mu)\mathbb{1}$, so it is not coupled with the other fields, and for instance, there is no Weinberg angle.

Nevertheless, the bosonic sector of the standard model can be reproduced in this model by introducing a more flexible metric tensor whose components still commute among themselves. In the case $n = 2$, this new metric tensor may be defined as

$$K_{kl} = \delta_{kl}(1 + \lambda\sigma_3) \tag{292}$$

such that the new dimensionless parameter λ can be related to the Weinberg angle. The symmetry breaking is achieved because now the kinetic part of the Higgs field Lagrangian $\frac{1}{2}\mathrm{Tr}[F_{\mu k}F^{\mu k}]$ is proportional to

$$(\nabla_\mu A_k^1)(\nabla^\mu A^{k,1}) + (\nabla_\mu A_k^2)(\nabla^\mu A^{k,2}) + (\partial_\mu A_k^0)(\partial^\mu A^{k,0}) + (\nabla_\mu A_k^3)(\nabla^\mu A^{k,3}) + 2\lambda(\partial_\mu A_k^0)(\nabla^\mu A^{k,3}) \tag{293}$$

Diagonalizing the last three terms is equivalent to the introduction of new linear combinations of components of the gauge field variables:

$$W_\mu^\pm = A_\mu^1 \pm iA_\mu^2 \tag{294a}$$

$$Z_\mu = gA_\mu^0 + g'A_\mu^3 \tag{294b}$$

i.e., to replacing $A_\mu = A_\mu^0 \otimes \mathbb{1} + A_\mu^k \otimes \sigma_k$ by

$$A_\mu = \begin{pmatrix} Z_\mu & W_\mu^- \\ W_\mu^+ & \tilde{A}_\mu \end{pmatrix} \tag{294c}$$

where

$$\tilde{A}_\mu = g'A_\mu^0 - gA_\mu^3 \tag{294d}$$

represents the photon field.

Around the vacuum $B_k^m = 0$ [i.e., $A_k^m = (i/\rho)\delta_k^m$], the gauge bosons $W_\mu^\pm$ and Z_μ have different masses, whereas $\tilde{A}_\mu$ is massless.

The quantity

$$\sin\theta_W = \frac{g}{(g^2 + g'^2)^{1/2}} \tag{295a}$$

is related to λ by quite a complicated relation:

$$\frac{1}{\cos^2\theta_W} = \frac{M_Z^2}{M_W^2} = \frac{3\lambda + \lambda^2}{1 + 3\lambda^2} \tag{295b}$$

with $|\lambda| < 1$. We recover a realistic value of θ_W for $\lambda = 0.181$.

This is for the simple model based on the module $C^\infty(\mathbf{R}^4) \otimes \mathbf{M}_2(C)$ with vacuum $\Omega_{1/2}$. In order to obtain more realistic models, one must look at other $C^\infty(\mathbf{R}^4) \otimes \mathbf{M}_2(C)$-modules. The next simplest right Hermitian $C^\infty(\mathbf{R}^4) \otimes \mathbf{M}_2(C)$-module is $C^\infty(\mathbf{R}^4) \otimes \mathbf{M}_{3\ 2}(C)$ (i.e., $K = 3$, $n = 2$). In this case, there are three vacua, Ω_0, Ω_1, and Ω_2, corresponding to the three inequivalent representations of $su(2)$ in C^2, namely $\{0\} \oplus \{0\} \oplus \{0\}$, $\{1/2\} \oplus \{0\}$, and $\{1\}$. We shall denote them by Ω_0, $\Omega_{1/2}$, and Ω_1, respectively.

Using the vacuum $\Omega_{1/2}$, i.e.,

$$\Omega_{1/2} \equiv \begin{pmatrix} i\sigma_k & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{296a}$$

one obtains a model close to the standard model in the bosonic sector by identifying appropriately the $U(1)$ part of the $U(1) \times SU(2)$ gauge potential and by making the field translations corresponding to $\Omega_{1/2}$:

$$A = \begin{pmatrix} iZ^k\sigma_k + iA^0\mathbb{1} & & -\overline{W}^1 \\ & & -\overline{W}^2 \\ W^1 & W^2 & -2iA^0 \end{pmatrix} = A_\mu d_1 x^\mu \tag{296b}$$

Nevertheless, this model has a defect because it contains too many bosonic fields. There is first a massless $U(1)$-gauge field A^0 which is completely decoupled and may be eliminated by introducing a generalization of a fiber volume for the module. Hence, one may add a global $U(1)$-gauge field $A' = A' \otimes 1_{3\times 3}$ which is completely decoupled. Second, there are two identical pairs of $W^{\pm}$ fields and two Z fields with

$$\frac{1}{\cos^2\theta_W} = \frac{M_Z^2}{M_W^2} = \frac{4}{3} \tag{296c}$$

It may be that this can be cured by considering some additional structure on the module to be conserved by the connections. It may be also that this is not a real defect.

7.3.3. A Tentative Introduction of Spinors

In the above section, we treated only the bosonic sector. To study the fermionic one, we have to define a noncommutative analog of the notion of a *spinor* (Kerner, 1990). This means that we would like to give a generalized meaning to the analog of Dirac's equation:

$$D\psi = i\gamma^{\mu}\nabla_{\mu}\psi = 0 \tag{297}$$

In order to give an appropriate noncommutative generalization of a spinorial field ψ, one has to define an A-module for which the Leibnitz rule holds [see equation (212b)]:

$$\nabla(\psi a) = (\nabla(\psi))a + \psi \otimes da \tag{298a}$$

with ψ belonging to the module that would generalize the module of sections of spinorial bundles in ordinary differential geometry, $a \in A$, and the exterior differential d acting on the algebra A.

For instance, consider the case $A = C^{\infty}(\mathbf{R}^4) \otimes \mathbf{M}_n(C)$. We should construct the generators of the Clifford algebra corresponding to the metric G [see equation (146)]:

$$G = g_{\mu\nu}\, d_1x^{\mu} \wedge d_1x^{\nu} + \rho^2 K_{kl}\theta^k \wedge \theta^l$$

with μ, $\nu = 0, 1, 2, 3$ and k, $l = 1, 2, \ldots, n^2 - 1$, i.e., construct the γ-matrices γ_{μ} and γ_k satisfying the relations

$$\nabla_{\mu}\gamma_{\nu} = 0 \tag{298b}$$

and, (298c)

$$\nabla_k(\bar{\psi}\gamma_m\psi) = (\nabla_k(\bar{\psi}))\gamma_m\psi + \bar{\psi}\gamma_m(\nabla_k(\psi))$$

Because $\bar{\psi}\psi$ should belong to A, it seems natural to define the derivation as

$$\partial_k \psi = -i\psi E_k \tag{298d}$$

and

$$\partial_k \bar{\psi} = iE_k \bar{\psi} \tag{298e}$$

so that

$$\begin{aligned} \partial_k(\bar{\psi}\psi) &= iE_k\bar{\psi}\psi - i\bar{\psi}\psi E_k = [iE_k, \bar{\psi}\psi] \\ &= \mathrm{ad}(iE_k)(\bar{\psi}\psi) = e_k(\bar{\psi}\psi) \end{aligned} \tag{298f}$$

Then, the covariant derivative should be generalized as follows:

$$\nabla_k = \partial_k + \frac{1}{8} C_{kl}^m \gamma_m \gamma^l \tag{298g}$$

The minimal module M satisfying all these relations is the following:

$$\psi \in M = C^{2^N} \otimes C^\infty(\mathbf{R}^4) \otimes C^n \tag{299}$$

with

$$N = \left[\frac{4 + n^2 - 1}{2}\right] \tag{300}$$

and $n \geq 2$.

In such a space, one can act with the γ-matrices and with the E_k-matrices which commute with γ-matrices. In general, one may choose Dirac's $4n \times 4n$ matrices γ_a with $a = (\mu, k)$, $\mu = 0, 1, 2, 3$, and $k = 1, 2, \ldots, n^2 - 1$, in the following way:

$$\gamma_a = \{\gamma_\mu \otimes \mathbb{1}, \gamma_5 \otimes E_k\} \tag{301a}$$

where γ_μ are the usual 4-dimensional Dirac matrices and $\{\mathbb{1}, E_k\}$ is the basis of $\mathbf{M}_n(C)$. The simplest module that can be considered is of course

$$C^\infty(R^4) \otimes M_{3\,2}(C) = C^{2^3} \otimes C^\infty(R^4) \otimes C^2 \tag{301b}$$

where the γ_a are defined as in (301a), with $\mathbb{1}$ being the 2×2 unit matrix and the E_k being the Pauli matrices σ_k [see equation (119)].

The minimal coupling with the gauge fields is ensured by replacing the covariant derivative ∇_k by a *gauge-invariant* one D_k given by

$$D_k = g\partial_k + \frac{i}{2} gE_k + \phi_k \tag{302}$$

where g is the overall scale factor, ϕ_k is a scalar field (Higgs field) representing the noncommutative part of the gauge potential [see equations (285)]

$$A_k = A_k^0 \otimes 1 + \phi_k = A_k^0 \otimes 1 + A_k^l \otimes E_l \tag{303a}$$

$$B_k = \mathrm{B}_k^0 \otimes 1 + B_k^l \otimes E_l \tag{303b}$$

$$B_k^0 = A_k^0 \tag{303c}$$

$$B_k^l = A_k^l - \frac{i}{\rho}\,\delta_k^l \tag{303d}$$

and E_k is given by

$$E_k = \frac{i}{\sqrt{2}}\,\sigma_k \tag{304}$$

At the symmetry-breakdown minimum, $B_k^l = 0$, the Dirac equation reads

$$\begin{aligned} D\psi &= i\gamma^a D_a\psi = [i(\gamma^\mu \otimes 1)\nabla_\mu]\psi + [i(\gamma^5 \otimes E_k)D_k]\psi \\ &\equiv i\gamma^\mu\nabla_\mu\psi + i\gamma^k D_k\psi = 0 \end{aligned} \tag{305}$$

The fermion masses will appear naturally if we find the *eigenvalues* of the *internal* Dirac operator

$$i\gamma^k D_k\psi = \mu g\psi \tag{306}$$

It is easy to find *eigenfunctions* of the form

$$\psi = \nu \otimes \xi \otimes \eta \tag{307}$$

where ν is a 4-component Dirac spinor over the space-time satisfying

$$\gamma^5\nu = \nu \tag{308}$$

$\xi \in C^2$ and $\eta \in \mathbf{M}_2(C)$. Then, the action of the *internal* Dirac operator reduces to

$$\begin{aligned} i\gamma^k D_k\psi &= \nu \otimes [E^k\xi \otimes (g\eta E_k + i\phi_k\eta) - \tfrac{1}{2}g(E_kE^k)\xi \otimes \eta] \\ &= \mu g\nu \otimes \xi \otimes \eta \end{aligned} \tag{309}$$

Around the broken symmetry phase, $A_k^l = (i/\rho)\delta_k^l$, one has

$$\phi_k = A_k^l \otimes E_l = \frac{i}{\rho}\,E_k = igE_k \tag{310}$$

and the eigenvalue equation (309) becomes

$$\sigma^k\xi \otimes [\sigma_k, \eta]_- + \tfrac{3}{2}\xi \otimes \eta = 2\mu\xi \otimes \eta \tag{311}$$

The solutions are found in the following way. If we denote

$$\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{312a}$$

and

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{312b}$$

then the following combinations are the eigenvectors of the *internal* Dirac operator:

$$\mu = \frac{3}{4}: \quad \psi_1 = \nu \otimes \xi_+ \otimes 1, \quad \psi_2 = \nu \otimes \xi_- \otimes 1 \tag{313a}$$

$$\mu = \frac{7}{4}: \quad \begin{cases} \psi_3 = \nu \otimes \xi_+ \otimes \sigma_+, & \psi_4 = \nu \otimes (\xi_+ \otimes \sigma_- + \xi_- \otimes \sigma_3) \\ \psi_5 = \nu \otimes \xi_+ \otimes \sigma_-, & \psi_6 = \nu \otimes (\xi_- \otimes \sigma_+ - \xi_+ \otimes \sigma_3) \end{cases} \tag{313b}$$

$$\mu = -\frac{5}{4}: \quad \begin{cases} \psi_7 = \nu \otimes \left(\xi_+ \otimes \sigma_- - \dfrac{1}{2}\, \xi_- \otimes \sigma_3 \right) \\ \psi_8 = \nu \otimes \left(\xi_- \otimes \sigma_+ + \dfrac{1}{2}\, \xi_+ \otimes \sigma_3 \right) \end{cases} \tag{313c}$$

Notice that the multiplicities 2, 4, and 2 of the three different eigenvalues correspond to the decomposition of the total space

$$C^{2^3} = C^8 = C^2 \otimes C^2 \otimes C^2$$

into the irreducible representations of $su(2)$ as $\frac{1}{2} \oplus \frac{3}{2} \oplus \frac{1}{2}$.

This minimal realization of spinors over the noncommutative geometry induced by the associative algebra $C^\infty(\mathbf{R}^4) \otimes \mathbf{M}_2(C)$ does not lead to the proper fermionic multiplets observed and realized in the standard model. The major shortcoming remains the absence of zero-mass modes for the fermions, as in classical Kaluza–Klein theories. Still, other realizations of noncommutative spinors may be more adequate.

8. CONCLUSION

The Dubois-Violette approach to noncommutative differential geometry has been presented in its whole in Sections 2, 4, and 6. Sections 3, 5, and 7 present applications.

This approach is based essentially on:

1. Replacing the commutative associative $C^\star$-algebra $A_0 = C^\infty(M)$ of the manifold M by an associative algebra A with unit.

2. Replacing the A_0-module of sections of a vector fiber bundle E over M by a right finite projective A-module.

3. Replacing the A_0-module Der(A_0) of derivations of A_0 by the algebra Der(A) of derivations of A, which is, in general, no longer an A-module.

4. Replacing the usual graded differential algebra $\Omega(M)$ of differential forms on M by the smallest graded differential subalgebra $\Omega_{\mathrm{Der}}(A)$ of the Chevalley complex $C(\mathrm{Der}(A); A)$, which contains A.

5. Defining a symplectic structure $\omega \in \Omega^2_{\mathrm{Der}}(A)$ for A and a generalized Poisson bracket which generalize the usual ones on the manifold M.

6. Defining on the right finite projective A-module a whole theory of connections (Hermitian structure, gauge, gauge transformation, connection, curvature, etc.), using the Connes formalism and the analogy between the A_0-module of sections of a vector fiber bundle E over M and abstract moduli. This construction led to the introduction of a notion of origin ∇_0 in the affine space of connections on the A-module, which plays a crucial role in the examples $A = \mathbf{M}_n(C)$ and $A = C^\infty(M) \otimes \mathbf{M}_n(C)$ presenting new models of field theory. Most of this review is essentially based on the work of Dubois-Violette (1988, 1989, 1990); Dubois-Violette *et al.* (1990a,b), Madore (1988, 1993a,b), and Kerner (1990).

It is evident that this approach may be generalized to the case where C is replaced by an arbitrary commutative field with vanishing characteristic.

Moreover, one may also replace, in the general formulation of this approach, Der(A) by any Lie subalgebra of Der(A). This approach can also be generalized to the case of a A-bimodules instead of right A-modules.

Furthermore, the example of $\mathbf{M}_n(C)$ (see Section 3.2) shows that $H_{\mathrm{Der}}(A)$ is not a Morita invariant, while, as is now established (see footnote 3), $H_{\mathrm{Out}}(A)$ is a Morita invariant. To show its link with the cyclic cohomology remains an open question.

In spite of the fact that the derivations of $\mathbf{M}_n(C)$ are inner, it is shown that one may develop a relatively rich differential geometric structure by using $\Omega_{\mathrm{Der}}(\mathbf{M}_n(C))$ as an algebra of differential forms.

The case of $A = \mathbf{M}_n(C)$ is very interesting because, in addition to its simplicity, it is purely noncommutative and corresponds to a typical quantum system of spin $s = (n - 1)/2$. It becomes clear that the quantum mechanics of a system with such a spin may be described in the framework of the noncommutative differential geometry of $\mathbf{M}_n(C)$.

The symplectic structure introduced on the algebra $\mathbf{M}_n(C)$ [or $A_0 \otimes \mathbf{M}_n(C)$] leads to a correspondence of the Poisson bracket of elements of A with $i/\hbar$ times their commutator. This represents the simplest realization of quantization.

It is worth noticing here that in quantum mechanics the derivations of the Heisenberg algebra $A_\hbar$ are also, in some sense, inner derivations, and that the discussion of Section 5.2 on symplectic structure is clearly relevant here.

In fact, it is shown that the quantum mechanics of spin systems and quantum systems of finite numbers of degrees of freedom is included in the framework of the generalization of symplectic geometry.

Furthermore, in the example of $\mathbf{M}_2(C)$ (see Section 3.2.5), the diagonalization of the operators Δ and $d + \delta$ gives discrete and finite eigenvalues that are to be compared with the infinite tower of excitations (e.g., Legendre polynomials) on a manifold S^2 when it plays the role of the *internal space* in Kaluza–Klein theories.

Although this approach is canonical and consistent, it presents some disadvantages. First, it is a more or less rigid. Second, the fact that Der(A) is not an A-module but only a module over the center $A_0 \otimes \mathbb{1}$ of A does not allows us to define a noncommutative analog of linear connections. So, in general, one cannot use the notion of connection of Section 6.3 for Der(A). Third, and for similar reasons, there is not yet a natural and general way to introduce the noncommutative analog of spinors.

Let us also remark that to study noncommutative symplectic geometry (see Sections 4 and 5), Dubois-Violette *et al.* have used explicitly all the details of the structure of the graded differential algebra $\Omega_{\mathrm{Der}}(A)$, i.e., the operation of Der(A) in $\Omega_{\mathrm{Der}}(A)$. But, to discuss new models of gauge theory, they used only the Z_2-grading of $\Omega_{\mathrm{Der}}(A)$ and the existence of the differential d. Replacing $\Omega_{\mathrm{Der}}(A)$ by a more general Z_2-graded differential algebra containing A, one arrives at models of gauge theory similar to those proposed by Connes and Lott (1989) and by Coquereaux *et al.* (1990).

Finally, for $A = C^\infty(M) \otimes \mathbf{M}_n(C)$ (see Sections 7.3.1 and 7.3.2), the proposed new models of gauge theory present some similarities with the bosonic part of the Weinberg–Salam model of electroweak interactions. The variables $A_k(x)$ play the role of Higgs fields and the sector Ω_1 is similar to the broken phase. One has then $U(1) \times SU(2)$ gauge theory and the mechanism that produces a mass for the $SU(2)$ part of the gauge field is very similar to the Higgs mechanism.

However, there are some differences. First, there are two stable gauge-invariant vacua corresponding to the gauge orbits of $\overset{0}{\nabla}$ and $\overset{1}{\nabla}$. Second, since the field variables A_k (or B_k) are the components of a Hermitian connection, they are anti-Hermitian and thus they do not interact with the electromagnetic field [i.e., with the $U(1)$ part A^0_μ of the A_μ]. Thus, there is nothing here like the Weinberg angle and the $U(1)$-gauge field is completely decoupled.

From the point of view of perturbation theory in $\mathbf{R}^4$, the gauge theories presented here are renormalizable. To carry out the renormalization program, one has to use the usual BRS technique. However, the standard BRS invariance does not forbid terms like $\mathrm{Tr}(A_k^2)$ with arbitrary coefficients. These would break the form of the action $S = \|\nabla^2\|^2$ given by equation (280) and one must therefore find an extended BRS or some other invariance that takes into account the fact that the action is a functional of a curvature. Another open question is to define a theory of spinors in this context.

Dubois-Violette *et al.* (1989a,b) gave an informal discussion of these models of gauge theory with a presentation of the analog of the scalar field for $A = C^\infty(\mathbf{R}^m) \otimes \mathbf{M}_n(C)$ and a discussion of the analogies and differences with Kaluza–Klein theories.

Finally, there are other approaches to noncommutative differential calculus and its applications (see, for instance, Connes, 1986; Karoubi, 1983; Woronowicz, 1987, 1989; Wess and Zumino, 1990; Zumino, 1991, 1992). The open questions raised in the Dubois-Violette approach and the link of the latter with the other approaches will be treated in a future work.

APPENDIX. REMARKS ON THE ALGEBRAIC CHARACTER OF THE PSEUDODIFFERENTIAL PROCESS

Let $A(C; +; \cdot)$ be an associative algebra over C and $\mathscr{L}(A)(A; \oplus; \mathrm{o})$ be the associative algebra of linear operators from A into A (Mourre, 1990).

Proposition 1. One defines on the vector space $A(C; +)$ a structure of associative algebra $A(C; +; \star)$ by means of the linear mapping

$$\Pi: \quad A(C; +) \to \mathscr{L}(A)$$

such that

$$\Pi(a) \circ \Pi(b) = \Pi(\Pi(a)(b))$$

and

$$a \star b = \Pi(a)(b)$$

$\forall a, b \in A$.

Definition A1. A *derivation* $D \in \mathscr{L}(A)$ of an algebra $A(C; +; \cdot)$ is a linear mapping from A into A such that

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$$

In general, one has the Leibnitz formula

$$D^{(n)}(a \cdot b) = \sum_{p=0}^{n} C_n^p D^{(p)}(a) \cdot D^{(n-p)}(b)$$

where $D^{(p)}(a)$ means the pth derivation of a and

$$C_n^p = \frac{p!}{n!(n-p)!}$$

Definition A2. Let A be an associative algebra equipped with a family of derivations $\{D_i\}$, $i \in I = \{1, 2, \ldots, n\}$ and a norm

$$\|\cdot\|: \quad A \to R^+$$

A is a complete metric topological algebra if one defines on it a metric

$$d(a, b) = d(0, a - b) = \rho(a - b)$$

where

$$\rho(a) = \|a\| + \sum_{\substack{\alpha \in \mathrm{N}^n \\ |\alpha| \geq 1}} \left(\frac{1}{2}\right)^{|\alpha|} \frac{\|D^\alpha(a)\|}{1 + \|D^\alpha(a)\|}$$

for any $a, b \in A$ and for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{N}^n$, with

$$D^\alpha = D_1^{\alpha_1} \cdot D_2^{\alpha_2} \cdots D_n^{\alpha_n}$$

and

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

Proposition 2. Let $A(C; +; \cdot)$ be an associative algebra equipped with a norm $\|\cdot\|$, $\{D_i\}_{i \in I}$ be a family of derivations of A that commute, and Π a linear mapping from A into $\mathcal{L}(A)$ such that, for any $D_1, D_2 \in (D_i)$, one has

$$\Pi(a) = a + (D_1(a))D_2 + \frac{1}{2!}(D_1^2(a))D_2^2 + \cdots + \frac{1}{n!}(D_1^n(a))D_2^n + \cdots$$

$$= \lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{n!}(D_1^n(a))D_2^n$$

Let $\mathrm{Def}(\Pi(a))$ be the domain of definition of $\Pi(a)$, i.e., the set of elements of A such that the above series converges for the metric topology.

Then, a subalgebra $\tilde{A}$ of $A(C; +; \cdot)$ exists such that:

1. $\tilde{A} \subset \mathrm{Def}(\Pi(a))$, $\tilde{a} \in \tilde{A}$, i.e., $\Pi(\tilde{A})(\tilde{A}) \subset \tilde{A}$.

2. The law $\tilde{a} \star \tilde{b} = \Pi(\tilde{a})(\tilde{b})$ induces on $\tilde{A}$ a structure of associative algebra $\tilde{A}(C; +; \star_{(D_1,D_2)})$:

$$\Pi(\tilde{a}) \circ \Pi(\tilde{b}) = \Pi(\Pi(\tilde{a})(\tilde{b}))$$

Moreover, it is clear that D_1 and D_2 remain derivations for the algebra $A(C; +; \star_{(D_1,D_2)})$.

Proposition 3. Let D be a derivation that commutes with D_1 and $D_2 \in \{D_i\}$. Then, D is also a derivation for the algebra $A(C; +; \star_{(D_1,D_2)})$, i.e.,

$$D(a \star b) = D(a) \star b + a \star D(b)$$

Proposition 4. Let $A(C; +; \cdot)$ be a normed associative algebra and let $\{D_1^i\}_{i\in I}$, $\{D_2^j\}_{j\in I}$, $I = \{1, \ldots, n\}$, be two families of commuting derivations:

$$[D_1^i, D_1^j] = [D_2^i, D_2^j] = [D_1^i, D_2^j] = 0$$

$\forall i, j \in I = \{1, \ldots, n\}$. For any multilabel $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, one has

$$\alpha! = \alpha_1!\, \alpha_2! \cdots \alpha_n!$$

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

$$D_1^\alpha = \prod_{i=1}^{n} (D_1^i)^{\alpha_i}, \qquad D_2^\alpha = \prod_{j=1}^{n} (D_2^j)^{\alpha_j}$$

Let

$$\Pi(a) = \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} (D_1^\alpha(a))D_2^\alpha$$

and let $\tilde{A}$ be the vector subspace introduced in Proposition 2; then the law

$$\tilde{a} \star_{(D_1,D_2)} \tilde{b} = \Pi(\tilde{a})(\tilde{b})$$

$\forall \tilde{a}, \tilde{b} \in \tilde{A}$ defines on $\tilde{A}$ a structure of associative algebra: $\tilde{A}(C; +; \star_{(D_1,D_2)})$.

Let now $A(C; +; \cdot)$ be an associative algebra, $\{D_1^i\}_{i\in I}$ and $\{D_2^j\}_{j\in I}$, $I = \{1, \ldots, n\}$, be two families of commuting derivations of A, and $\tilde{A}$ be the vector space introduced in Proposition 2. Let

$$A(C; +; \star_{(\cdot;\nu;\{D_1^i\},\{D_2^j\})}) \qquad \text{with} \quad \nu \in C$$

be a family of associative algebras defined by

$$\Pi_\nu(a) = \sum_{n\geq 0} \frac{\nu^n}{n!} (D_1^n(a))D_2^n$$

For $\nu = 1$, we recover the usual product introduced above, i.e.,

$$\star_{(\cdot;1;\{D_1^i\},\{D_2^j\})} = \star_1 \equiv \star_{(\{D_1^i\},\{D_2^j\})}$$

for any two families $\{D_1^i\}_{i\in I}$, $\{D_2^j\}_{j\in I}$ of commuting derivations.

Proposition 5. Consider the algebra obtained by the *deformation*

$$A(C;\, +;\, \star')$$

such that

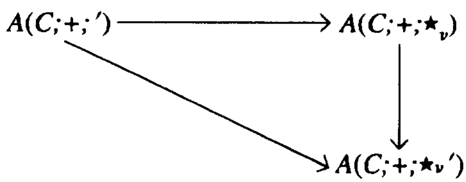
$$\star' = \star_{(\star_\nu;\nu';\{D_1^i\},\{D_2^j\})}$$

$$= \star_{(\star_{(\cdot\,;\nu;\{D_1^i\},\{D_2^j\})};\nu';\{D_1^i\},\{D_2^j\})}$$

Then

$$\star' = \star_{\nu+\nu'} = \star_{(\cdot\,;\nu+\nu';\{D_1^i\},\{D_2^j\})}$$

i.e., the following diagram commutes:



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